Clearing Algorithms and Network Centrality

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Summary & Main Findings

- I show that the solution of a standard clearing model used in contagion analyses can be expressed as a generalized form of a Katz centrality measure under certain conditions
- These conditions mostly have a clear economic interpretation, most importantly the occurrence of a shock that renders all institutions insolvent
- This finding provides a formal explanation for the previously empirically observed close relation between the Katz centrality and contagiousness
- This finding allows to analyze the assumptions that one is making when using centrality measures for systemic risk analyses. I argue that they should be considered too strong.

Clearing Model

The standard clearing framework introduced by Eisenberg and Noe (2001) is based on a balance sheet framework:

Figure 1: Stylized balance sheet

Interbank Assets $(Cl)_i$ (or $(Cp)_i$)	Equity $a_i + (Cp)_i - l_i$
Other assets a_i	Liabilities l_i

Definition	[Computation]	Description
$N \in \mathbb{N}$		Number of banks in the system plus one (sink node)
$a \in \mathbb{R}_+^N$		External assets of bank i
$L \in \mathbb{R}_+^{N \times N}$		L_{ij} = liabilities of i towards j
$l \in \mathbb{R}_+^N$	$l_i = \sum_j L_{ij}$	Total (nominal) liabilities of i
$p \in \mathbb{R}_+^N$	$f(p) = p$	Clearing payment vector of payments that are actually made (as opposed to nominal liabilities)
$C \in [0,1]^{N \times N}$	$C_{ij} = \begin{cases} \frac{L_{ij}}{l_j} & \text{if } l_j > 0 \\ 0 & \text{otherwise} \end{cases}$	If $C_{ij} > 0$, it represents the relative share that bank i's claim on bank j has among the total liabilities of bank j.
$D \in \{0,1\}^{N \times N}$	$D_{ij}(x) = \begin{cases} 1 & \text{if } i = j \wedge \\ & a_i + (Cx)_i < l_i \\ 1 & \text{if } i = j = N \\ 0 & \text{otherwise} \end{cases}$	If $D(x)_{ii} = 1$, i is in default under payment vector x. The sink node is always set to default.
$r \in [0,1]$		Recovery rate for assets

Which leads to a convenient expression for the **balance sheet equation**:

$$Equity = Assets - Liabilities = a + Cp - l$$

- C multiplied with any vector by construction gives the asset value of corresponding payments for each bank
- Cp gives the value of all interbank claims assuming that solvent banks fully repay their liabilities and insolvent banks repay the remaining value of their assets
- p is called the clearing payment vector and computed as the fixed point of the following map (setup by Rogers and Veraart (2012)):

$$f(p) = \overbrace{(I - D(p))\,l}^{\text{Solvent banks repay in full}} + D(p)\overbrace{\left(r_a a + rC\left(\underbrace{D(p)\,f(p)}_{\text{Claims on insolvent banks}} + \underbrace{(I - D(p))\,l}_{\text{Claims on solvent banks}}\right)\right)}^{\text{Insolvent banks repay remaining asset value}} \quad (1)$$

Katz Centrality

The Katz centrality measure is based on the following intuition:

- Every node starts with a start weight β (usually set equal to **1**)
- Every node receives the weight from all its neighbours multiplied by a "dampening" factor $\alpha \in (0,1)$

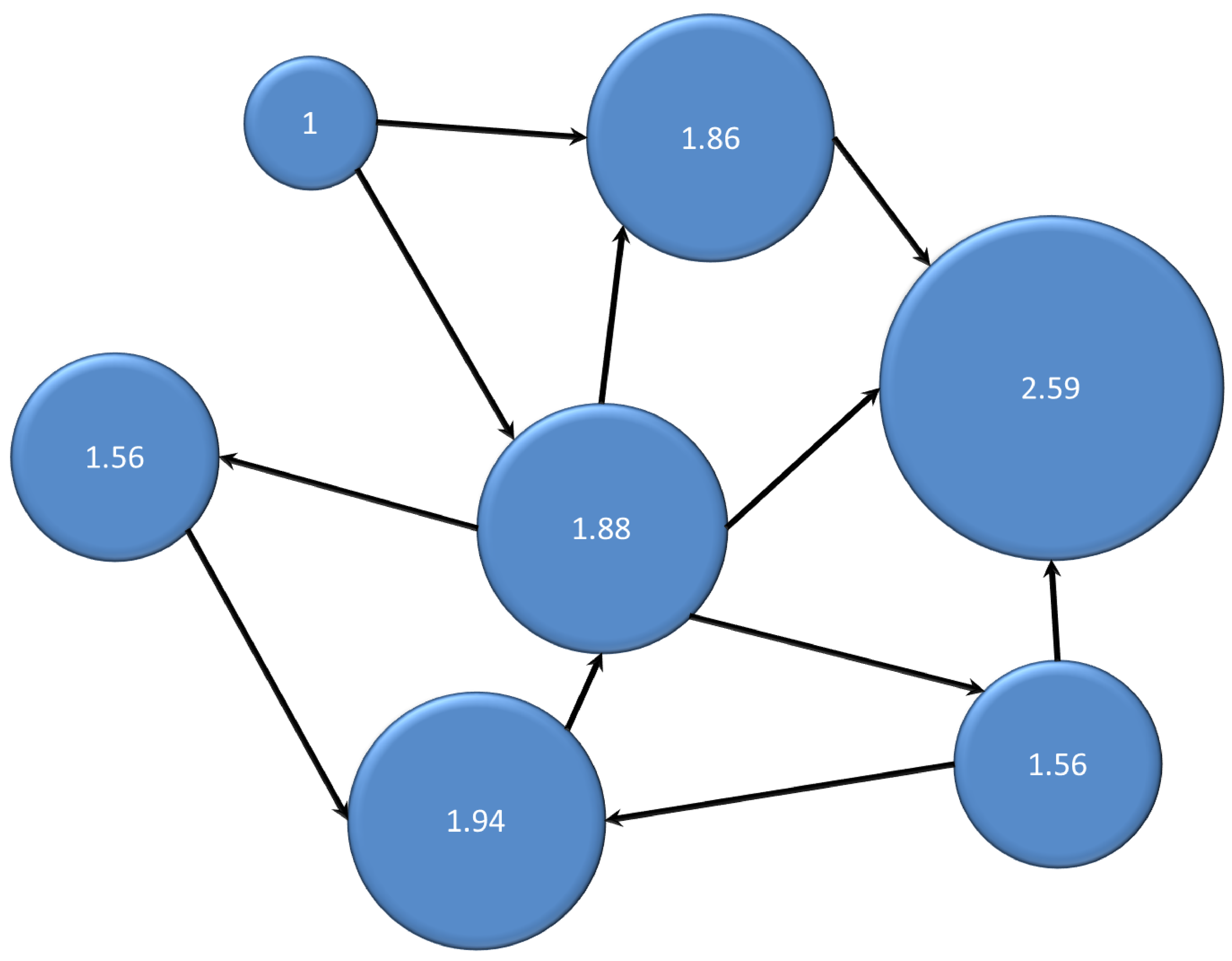
Let A be the adjacency matrix of a graph, then the Katz centrality of node i can be expressed as:

$$x_i = \alpha \sum_k A_{k,i}\, x_k + \beta$$

Which for $\alpha \neq \frac{1}{\rho(A)}$ can be solved in matrix form:

$$x = \sum_{i=0}^{\infty} (\alpha A)^i \beta = (I - \alpha A')^{-1} \beta$$

Figure 2: Katz Centrality example ($\alpha = 0.3, \beta = \mathbf{1}$)



Equivalence

Under the conditions specified hereafter, a systemic risk measure $\sigma = l - p$ based on the solution to equation 1 can be expressed as:

$$\sigma = (I - rC)^{-1} \beta \quad (2)$$

With $\beta_i = (1-m)l_i - (r-m)(Cl)_i \forall r, m \in (0,1), i < N$, where $m \in (0,1)$ is an interpolation factor.

The main conditions for equation 2 to hold are:

- All banks in the system are insolvent
- External assets are greater than equity for all banks
- There has to exist an outside world to which liabilities exist