A Zero Phase Shift Band Pass Filter

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Introduction

- Band pass filtering is a powerful tool to decompose time series on the basis of frequency characteristics without strong model assumptions
- Ideal band pass filter: suppress all fluctuations outside pass band, leave all frequencies within pass band unaltered, induces no phase shifts
- Ideal band pass filtering requires infinite data
- Well known finite sample time domain approximations: Baxter-King (1999), Christiano-Fitzgerald (2003)
- Alternative: direct frequency filter on Fourier transformation
- We propose direct frequency filter combined with iteratively fitted trigonometric functions, largely based on Bloomfield (1976) and Schmidt (1984)

Ideal band pass filter

- Linear filter:

$$G(L) = \sum_{k=a}^{b} g_k L^k$$

applied to series x_t gives filtered series $y_t = G(L)x_t$

- From the frequency response function (FRF)

$$G(e^{-i\omega}) = \sum_{k=a}^{b} g_k e^{-i\omega k}$$

we obtain

$$Gain(\omega) = |G(e^{-i\omega})|$$

and

$$Phase(\omega) = \arg(G(e^{-i\omega}))/(2\pi)$$

Ideal band pass filter — cont.

- Ideal band pass filter has

$$Gain(\omega) = \begin{cases} 1 & \text{for } \omega_1 \leq \omega \leq \omega_2 \\ 0 & \text{for } \omega < \omega_1 \text{ or } \omega > \omega_2 \end{cases}$$

and

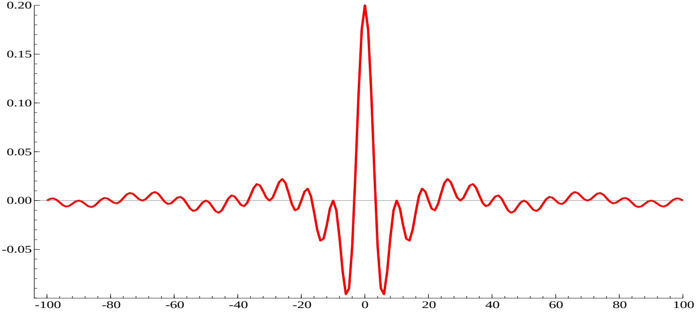
$$Phase(\omega) = 0, \quad \omega \in [0, \pi]$$

Ideal band pass filter — cont.

- Ideal filter weights in time domain are

$$g_k = \begin{cases} \frac{\sin(\omega_2 k) - \sin(\omega_1 k)}{\pi k} & \text{for } k \neq 0 \\ \frac{\omega_2 - \omega_1}{\pi} & \text{for } k = 0 \end{cases}$$

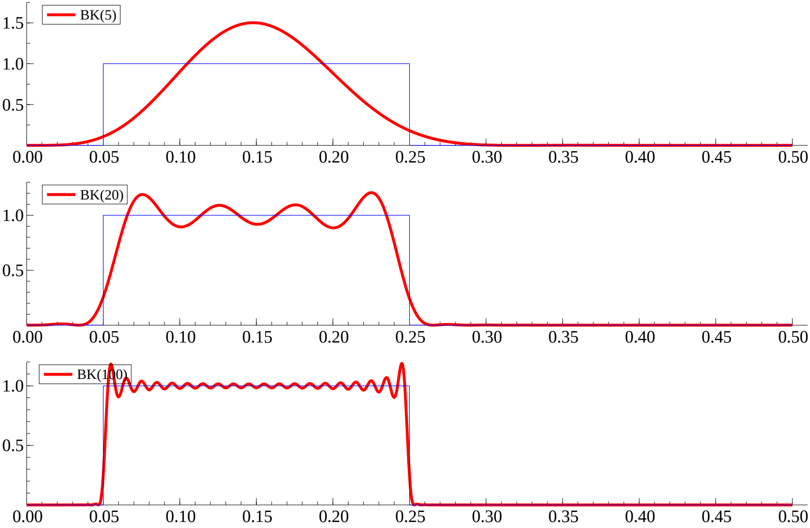
- Weights are still substantial at long lags



Baxter-King and Christiano-Fitzgerald filter

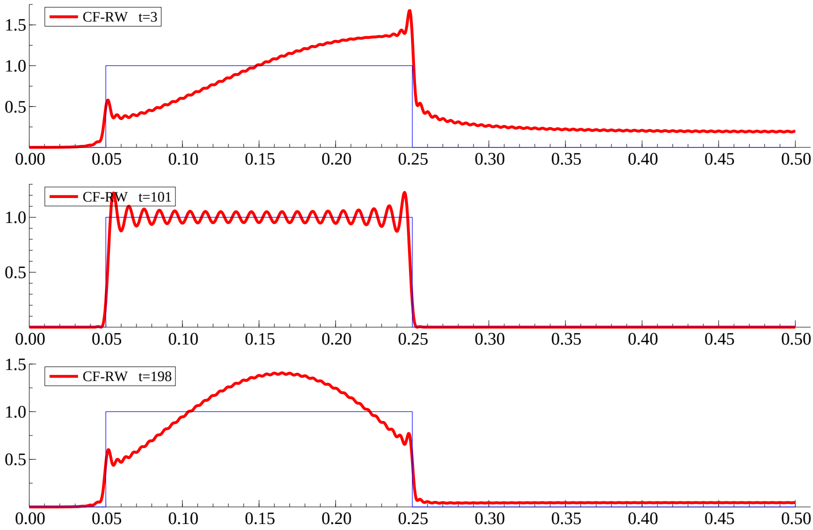
- Baxter-King: truncate weights beyond lag a, impose constraint at frequency $\omega = 0$ (detrending)
  - Symmetric (no phase shift)
  - Removes first and second order trends
  - No output for first and last a observations
- Christiano-Fitzgerald: extrapolate data sample using assumed model, adjust end point weights $\tilde{g}_k$ (dependent on assumed model for x_t)
  - Filter (and therefore also Gain) is time varying; end point adjustments for y_t depend on t
  - Weights are asymmetric, so filter is not phase-neutral
  - Requires model assumption for x_t (recommended: Random Walk)

Baxter-King Squared Gain



(cycles per period)

Christiano-Fitzgerald Squared Gain



(cycles per period), T=201

Direct Frequency Filter

1. Calculate the discrete Fourier transform (DFT)

$$J_j = \frac{1}{T} \sum_{t=0}^{T-1} x_t e^{-i\omega_j t}, \qquad \omega_j = 2\pi j/T,$$

   for $j = 0, \ldots, T-1$.
2. Multiply the Fourier coefficients J_j with the desired frequency response function to obtain $\tilde{J}_j$.
3. Apply the inverse discrete Fourier transform (IDFT)

$$x_j = \sum_{j=0}^{T-1} \tilde{J}_j \exp(i\omega_j t)$$

   on $\tilde{J}_t$ to transform the series back into the time domain.

Direct Frequency Filter – cont.

- DFT on finite data has errors due to *leakage*
- Errors are worse for smaller samples, and for data with higher peaks in spectrum/periodogram
- Upper bound for filter error due to leakage:

$$|y_t - \hat{y}_t|^2 \leq M^2 \Big(\sum_{k \in Q} |g_k| \Big)^2,$$

$$Q = \{-\infty, \ldots, t - T\} \cup \{t + 1, \ldots, \infty\},$$

where y_t is ideal filtered series, $\hat{y}_t$ is finite sample direct frequency filtered series,

$$M = \max_{0 \leq \omega \leq 2\pi} |T \cdot J(\omega)|$$

$J(\omega)$ is continuous extension of Fourier transform

Zero Phase Filter

- Direct frequency filter error can be reduced by increasing sample (usually impossible), or reducing maximum periodogram (possible)
- Decompose series in fitted trigonometric series and remainder

$$x_t = \alpha \cos(\theta t) + \beta \sin(\theta t) + r_t$$

- Trigonometric component can be filtered perfectly (avoiding leakage), and remainder with smaller error than directly filtering x_t
- Use least squares to minimise:

$$V(\alpha, \beta, \theta) = \sum_{t=0}^{T-1} \Big(x_t - \alpha \cos(\theta t) - \beta \sin(\theta t) \Big)^2$$

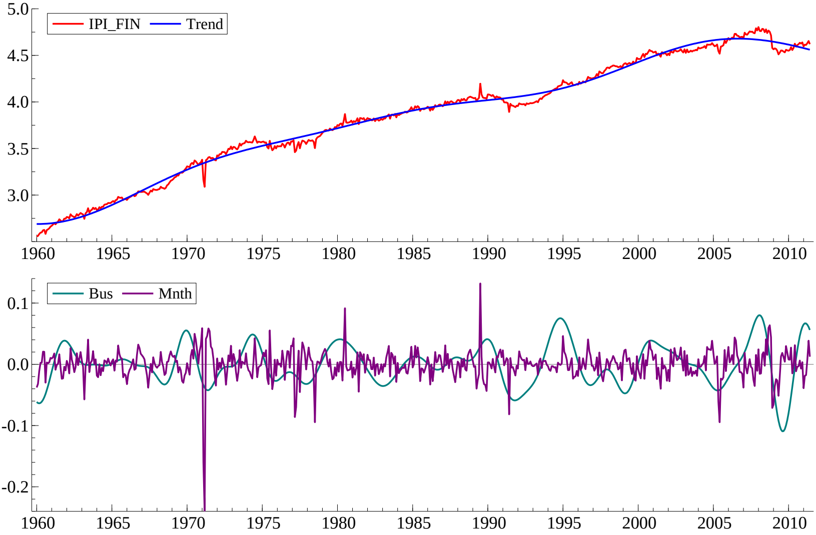
Zero Phase Filter — cont.

- Sum of squared errors is proportional to Fourier transform, which gives upper bound to (leakage induced) filter error.
- Optimisation is linear for given θ, non-linear in general
- Objective function has many local optimums, so we use combination of grid evaluation of $\theta \in [0, \pi]$ and Brent's method
- Extensions:
  - Include constant term μ
  - Fit multiple trigonometric series simultaneously
  - Fit new trigonometric series on r_t, iterate until final remainder is very small
- Final filtered series is sum of fitted trigonometric series with frequencies θ_j within the pass band plus direct frequency filtered final remainder

Zero Phase Filter — cont.

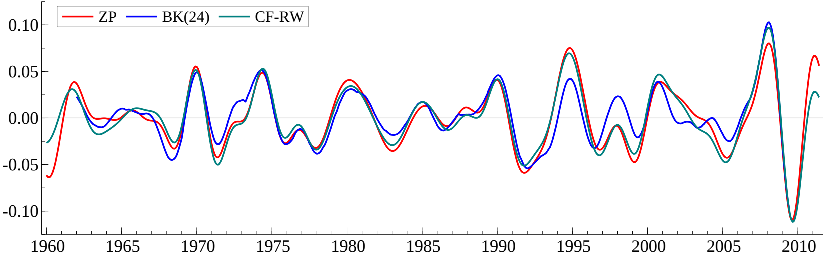
- Choices of number of simultaneous components to fit and stopping criteria for iterations based on simulation and extensive experience
- Majority of the filtering takes place in time domain, based on trigonometric fit, and is filtered with exact pass band
- Leakage error in remainder can be made very small by increasing iterations
- Filter induces no phase shift and has no irregularities near end points
- Disadvantage: calculation time is much higher than time domain approximations or direct frequency filtering, so efficient implementation required

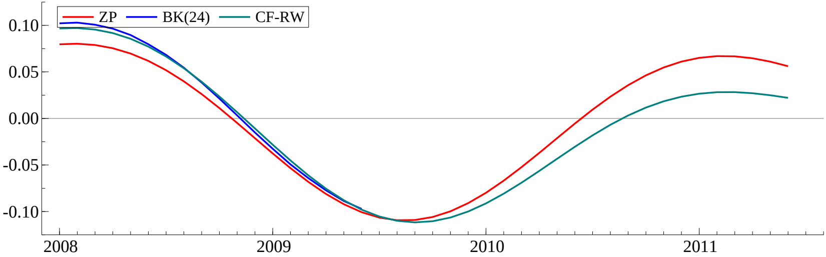
Illustration: Industrial Production Index Finland



Trend: (> 8 year), Bus (2–8 year), Mnth (< 2 year)

Illustration: Industrial Production Index Finland

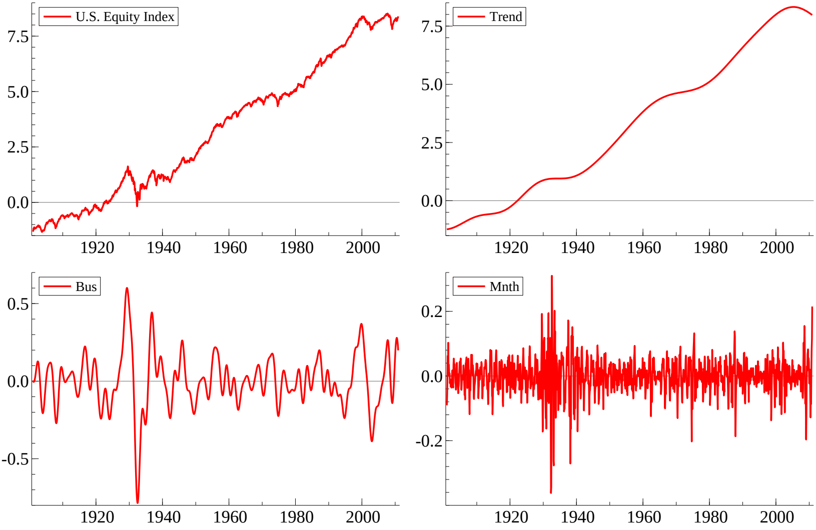




Multi-period returns

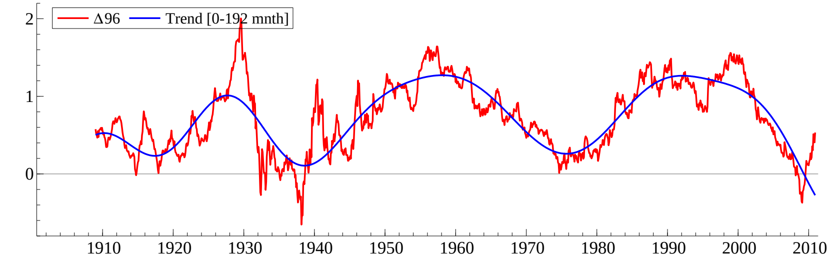
- Stock index fluctuation: dominated by trend, but much medium and high frequency movements
- Direct band pass filter decomposes the periodic movements
- Multi-period log-returns (e.g., returns at 8 year, 1 year, 1 month horizons) show similar dynamics
- Long horizon series also contain medium and short period fluctuations

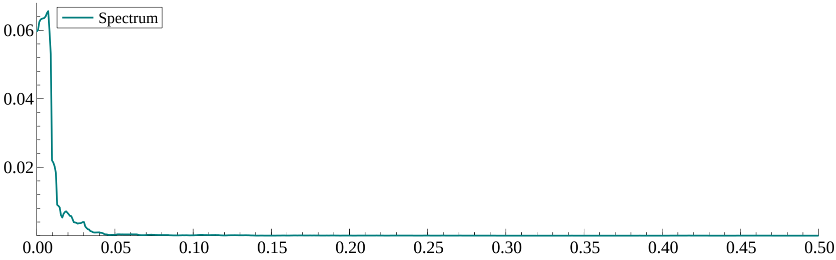
Multi-period returns — cont.



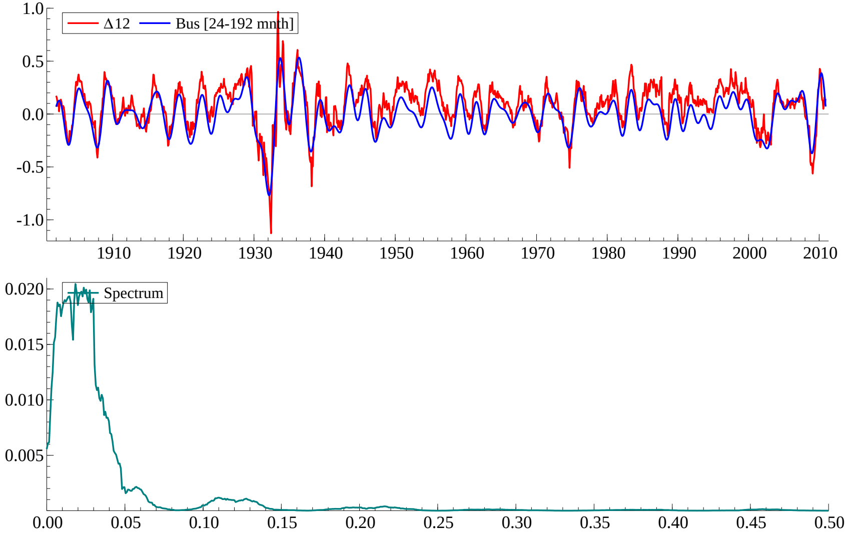
Trend (> 16 year), Bus (2–16 year), Mnth (< 2 year)

Multi-period returns — cont.

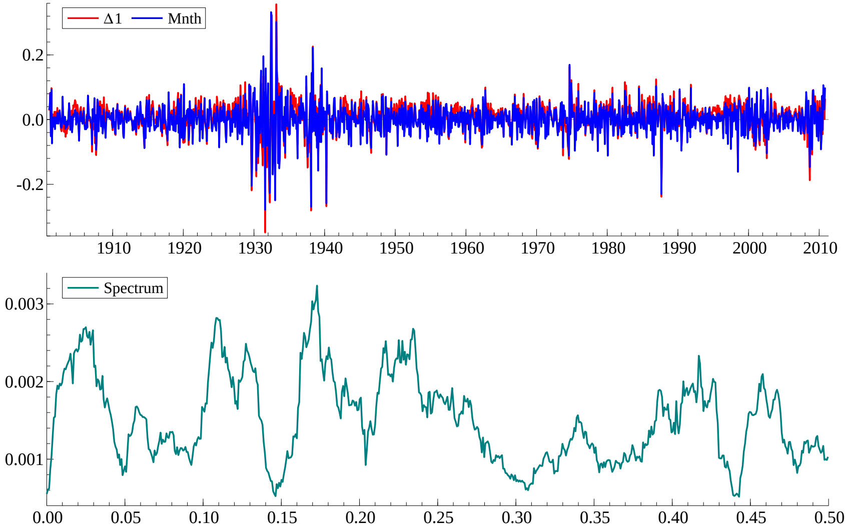




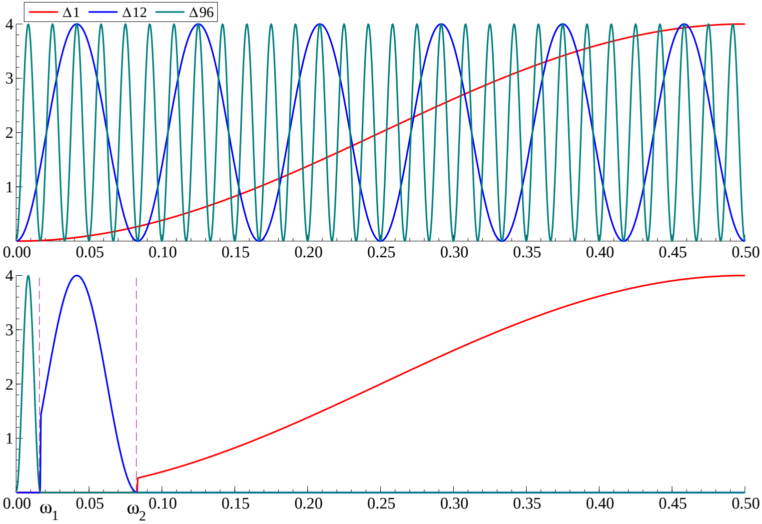
Multi-period returns — cont.



Multi-period returns — cont.



Multi-period returns — cont.

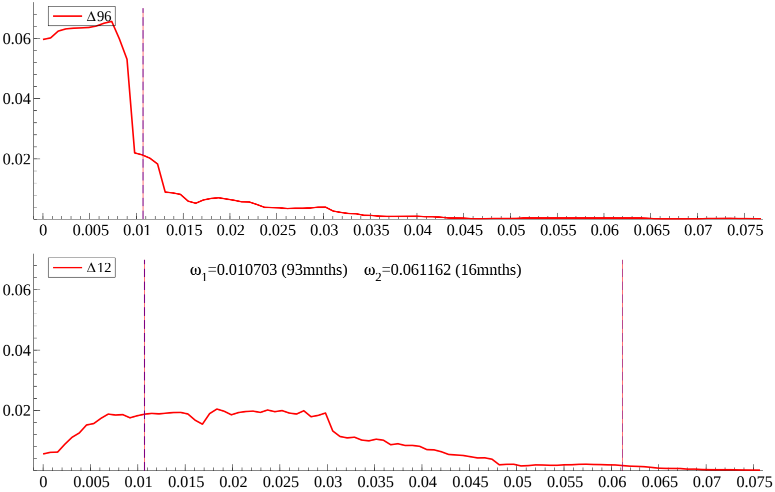


Squared gain of multi-period (seasonal) differencing (top), and band pass (ω_1 to ω_2) filtering multi-period difference (bottom)

Multi-period returns

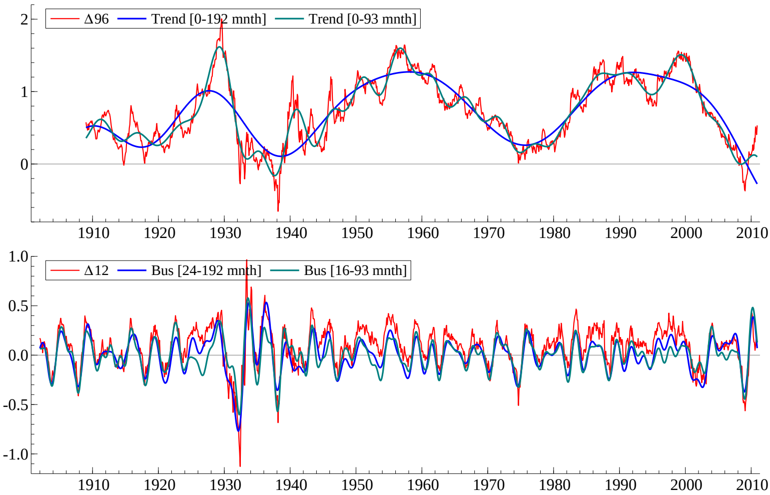
- We want to perform a frequency decomposition such that taking 8 year, 1 year 1 month returns on the decomposition matches 8 year, 1 year 1 month returns as closely as possible
- In time domain: filter with different pass bands and minimise squared error on all three horizons
- In frequency domain: maximise sum of spectra of filtered returns
- Results in proper frequency decomposition of the entire series that show the dominant dynamics at the horizons of interest
- Proper decomposition makes it possible to make separate models for different frequencies that can be summed to a model for the original series

Multi-period returns — cont.

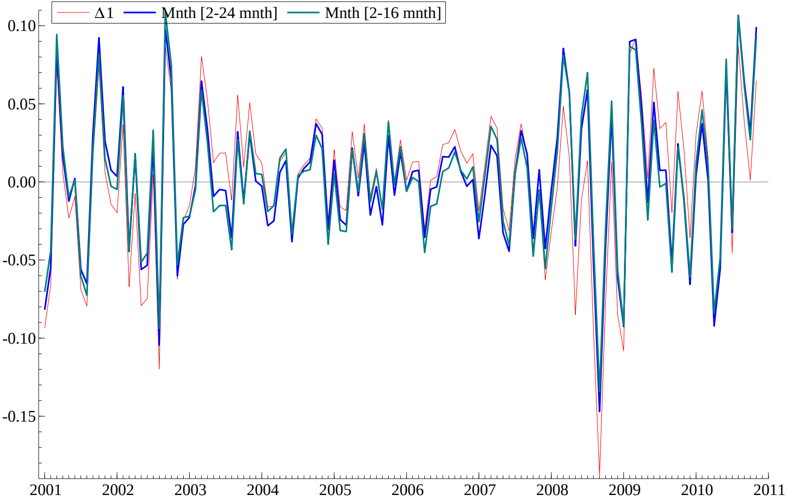


Optimised pass bands

Multi-period returns — cont.



Multi-period returns — cont.



Conclusions

- Time domain approximations of ideal frequency filter are inaccurate for small samples, especially near endpoints
- Zero Phase filter: fit multiple trigonometric functions to data, apply direct frequency filter on remainder
- Exact pass band applies to trigonometric components, leakage errors on small remainder
- Output is similar to CF in middle part, differences near the end
- Financial application: given interest for returns at different horizons, pass bands can be chosen such that filtered series retain dynamics of interest at all horizons

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