Risk-sharing or risk-taking? Counterparty risk, incentives and margins*

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Abstract

We analyze optimal hedging contracts between a protection buyer and protection sellers. When a seller observes that the hedge is likely to become a liability for her, it reduces her incentives to exert unobservable risk-prevention effort. Hedging can thus generate endogenous counterparty risk. To avoid such risk, the protection buyer may prefer to give up some insurance to maintain the seller's incentives. Variation margins, compelling the protection seller to liquidate some of her risky positions after bad news, relax the incentive problem. Initial margins address the market failure caused by unregulated trading of hedging contracts among protection sellers.

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1 Introduction

The development of derivative markets such as forwards, futures or credit default swaps (CDS) can enhance risk-sharing opportunities. Yet, as noted by Rajan (2006), it can also induce greater risk-taking. We study this tradeoff in a moral hazard context and show how it limits the scope for risk-sharing and how it can lead to endogenous counterparty risk. We analyze how margin deposits align incentives and improve risk-sharing.

Consider a financial institution whose assets (e.g., corporate or real-estate loans) are exposed to risk. Due to leverage or regulatory constraints, such as risk-weighted capital requirements, the institution would benefit from hedging its risk. To do so, the financial institution contacts a protection seller, e.g., an insurance company or another financial institution, and the two parties design an optimal risk-sharing contract.

Before engaging in that derivative trade, the protection seller already has assets in place. To reduce the downside risk on her assets, the protection seller must exert effort. For example, she must acquire information to screen out bad loans, or she must monitor borrowers. As in Thompson (2010), we assume there is a moral hazard problem on the side of the protection seller. More precisely, we assume that, while costly, the risk-prevention effort of the limited-liability protection seller is not observable.[1]

Ex-ante, when the protection seller enters the position, the derivative contract is neither an asset nor a liability. For example, the seller of a credit default swap pays the buyer in case of credit events (default, restructuring) but collects an insurance premium otherwise, and on average these costs and benefits offset each other. But, when the protection seller observes bad news about the underlying asset of the derivative trade, the trade becomes a liability for her. For example, on observing a strong drop in real estate prices, sellers of subprime-mortgage CDS anticipate the positions to move against them so that they would have to make insurance payments.

The liability embedded in the derivative trade undermines the incentives of the protection seller to exert effort to reduce the downside risk of her other assets.[2] Similar to the debt overhang effect analyzed by Myers (1977), the protection seller bears the full cost of such

[1]In most of our analysis, the unobservable action of the agent affects the cash–flows in the sense of first–order stochastic dominance, as in Holmstrom and Tirole (1998) and Tirole (2005). Yet, we show that qualitatively identical results hold if the unobservable action leads to an increase in risk in the sense of second–order stochastic dominance, in the spirit of Jensen and Meckling (1976).

[2]If the balance sheet of the protection seller is not marked to market, it does not reflect this liability. Regulation should be aware of such hidden liabilities in the derivative positions of financial institutions.

efforts while part of its benefits accrue to the protection buyer.[3] To preserve the seller risk-prevention incentives, the protection buyer can accept a smaller insurance after bad news to reduce the liability of the protection seller. Or if such reduced insurance is too costly ex-ante, the protection buyer may prefer to give up on the seller's incentives after bad news. By no longer exerting risk-prevention effort, the seller runs the risk of default.[4] When the default of the protection seller occurs, it generates counterparty risk for the protection buyer.[5]

Our analysis thus identifies a channel through which derivative trading can propagate risk. Without moral hazard, the risk exposures of the protection buyer and seller are independent. Moral hazard can, however, lead to a lack of risk-prevention effort, or risk-taking, and the default of the seller after bad news about the risk of the protection buyer.[6]

The optimal hedging contract stipulates the circumstances under which the protection seller must liquidate a fraction of her risky assets and deposit the resulting cash on a margin account. The cost of such liquidation is the wedge between what the assets could have earned under the seller's effort and the lower risk-free return on cash in a margin deposit. The benefit is that the cash in the margin account is safe and is no longer under the control of the protection seller. It is ring-fenced from moral hazard. We show that calling margins after bad news, i.e., requiring variation margins, relaxes the moral hazard problem and increases incentive-compatible insurance. The overall effect of margins on risk is, however, ambiguous. Since margin deposits can be used to pay the protection buyer when the seller defaults, they reduce the incentives of the buyer to deter the seller's risk-taking.

We extend the analysis to the case of multiple sellers.[7] When the sellers can retrade, fully transferring risk exposure among themselves, the equilibrium differs from the information constrained second-best. This offers a rationale for the regulation of retrading among CDS protection sellers ("novation") discussed in Duffie et al. (2010).

As mentioned above, our analysis is in line with Thompson (2010) since in both models there is moral hazard on the part of the protection seller. But Thompson (2010) analyzes

[3]Note however that instead of exogenous debt as in Myers (1977) our model involves endogenous liabilities pinned down by optimal contracting.

[4]Stephens and Thompson (2011) also analyze the risk of insurer default, but in a model with good and bad insurers.

[5]For example, Lehman Brothers and Bear Stearns defaulted on their CDS derivative obligations because of losses incurred on their other investments, in particular sub-prime mortgages.

[6]This incentive-based theory of propagation differs from the analyses of systemic risk offered by Freixas, Parigi and Rochet (2000), Cifuentes, Shin and Ferrucci (2005), and Allen and Carletti (2006).

[7]Stephens and Thompson (2011) also analyze competition among multiple protection sellers. While we study retrading among protection sellers, they analyze the role of exclusivity and non-exclusivity, respectively.

the case in which the protection buyer is privately informed about his type and shows that moral hazard can alleviate this adverse selection problem. In contrast, there is no adverse selection in our analysis and we identify a different channel through which moral hazard reduces insurance and generates counterparty risk. Other important contributions to these issues include Acharya and Bisin (2010), who focus on the contractual externality between protection buyers and the role of transparent centralized clearing in this context, and Bolton and Oehmke (2011) who argue that derivatives should not be senior in bankruptcy relative to other creditors.

The model is presented in the next section. In Section 3, we analyze the benchmark case in which effort is observable. Then, we turn to optimal contracting under moral hazard. To highlight the basic trade-off between risk-sharing and risk-taking, we first abstract from margins in Section 4. In Section 5, we analyze the optimal contract with margins. In Section 6, we analyze the case of multiple protection sellers. Section 7 concludes. Proofs are in the Appendix.

2 The model

There are three dates, $t = 0, 1, 2$, and two agents, the protection buyer and the protection seller, who can enter a risk-sharing contract at $t = 0$.

Players and assets: The protection buyer is risk-averse with twice differentiable concave utility function, denoted by u. At $t = 0$ he is endowed with illiquid risky assets with random return $\tilde{\theta}$ realized at $t = 2$. For simplicity, we assume that $\tilde{\theta}$ can take on two values: $\bar{\theta}$ with probability π and $\underline{\theta}$ with probability $1 - \pi$. The protection buyer seeks insurance against the risk $\tilde{\theta}$. The protection seller is risk-neutral. At time $t = 0$ she has an amount $A > 0$ of assets in place that have an uncertain per unit return $\tilde{R}$ at $t = 2$. To the extent that $\tilde{R}$ is random the balance sheet of the protection seller is risky.

The protection buyer could be a commercial bank seeking to hedge the credit risk of its industrial or real estate loan portfolio.[8] The protection seller could be an investment bank or an insurance company.[9]

[8]Concavity of the objective function of the protection buyer can reflect institutional, financial or regulatory constraints, such as leverage constraints or risk–weighted capital requirements. For an explicit modeling of hedging motives see Froot, Scharfstein and Stein (1993) and Froot and Stein (1998).

[9]A prominent example is AIG. 72% of the CDS it had sold by December 2007 were used by banks for capital relief (European Central Bank, 2009).

At $t = 1$ the protection seller makes unobservable decisions affecting the riskiness of her assets. To capture the moral hazard problem in the simplest possible way, we follow Holmstrom and Tirole (1998) and Tirole (2005), and assume she can choose between effort, $e = 1$, and no effort, $e = 0$. Exerting effort is unobservable and leads to an improvement in $\tilde{R}$ in the sense of first order stochastic dominance. We assume that in this case the return realized at time 2 is deterministic and equal to AR. Alternatively, if the protection seller shirks and does not exert effort, this exposes her investments to downside risk, and the return on her assets is equal to AR with probability p and 0 with probability $1 - p$. Shirking, however, gives the protection seller a private benefit B per unit of assets on her balance sheet. Equivalently, the private benefit of shirking can be interpreted as the cost of effort. The protection seller has limited liability. When her assets yield 0, she cannot make any payment promised to the protection buyer, who is therefore exposed to counterparty risk. This environment is meant to capture the essence of controlling risk in a financial institution. When exerting effort, the protection seller spends resources to carefully monitor her investments and thus avoid a large risk of default. When she shirks on effort, e.g., by relying on easily available but superficial information such as ready-made ratings, she exposes herself to the risk of default. Hence, we refer to shirking also as risk-taking.

We normalize the discount factor to one and assume that

$$R > 1 \text{ and } R > pR + B.$$

The first inequality implies that under effort the assets of the protection seller are a positive NPV project. The second one implies that shirking destroys value. Hence, if the protection seller does not enter into a contract with the buyer and is solely concerned with managing her assets, she prefers to exert effort. Finally note that for a given level of effort, $\tilde{R}$ and $\tilde{\theta}$ are independent.

Advance information: A public signal $\tilde{s}$ about $\tilde{\theta}$ is observed at $t = 0.5$, before the seller makes her effort decision at $t = 1$. For example, when $\tilde{\theta}$ is the value of a real estate loan portfolio held by the protection buyer, $\tilde{s}$ is the value of a real estate index or an indicator of default risk for these assets. Denote

$$\lambda = \text{prob}[\bar{s}|\bar{\theta}] = \text{prob}[\underline{s}|\underline{\theta}].$$

The probability π is updated to $\bar{\pi}$ upon observing $\bar{s}$ and to $\underline{\pi}$ upon observing $\underline{s}$, where, by Bayes' law,

$$\bar{\pi} = \text{prob}[\bar{\theta}|\bar{s}] = \frac{\lambda\pi}{\lambda\pi + (1-\lambda)(1-\pi)} \text{ and } \underline{\pi} = \text{prob}[\bar{\theta}|\underline{s}] = \frac{(1-\lambda)\pi}{(1-\lambda)\pi + \lambda(1-\pi)}.$$

We assume that $\lambda \geq \frac{1}{2}$. If $\lambda = \frac{1}{2}$, then $\bar{\pi} = \pi = \underline{\pi}$ and the signal is completely uninformative. If $\lambda > \frac{1}{2}$, then $\bar{\pi} > \pi > \underline{\pi}$, i.e., observing $\bar{s}$ increases the probability of $\bar{\theta}$ (good signal) whereas observing $\underline{s}$ decreases the probability of $\bar{\theta}$ (bad signal). If $\lambda = 1$, then the signal is perfectly informative.

Margins: The protection seller can liquidate a fraction α of her assets and deposit the resulting cash on a margin account. The cost of such deposits is that their value at time 2, αA, is lower than what it could have been had the assets remained under the management of the protection seller, αAR.

Yet margins also have advantages. Our key assumption is that the cash deposited in the margin account is safe and no longer under the discretion of the protection seller, i.e., it is ring-fenced from moral hazard. Furthermore, if the protection seller defaults, the cash on the margin account can be used to pay the protection buyer.

Margin accounts can be implemented as escrow accounts set up by the protection buyer or via a market infrastructure such as a central counterparty (CCP). Importantly, we assume that margin deposits are observable and contractible, and that contractual provisions calling for margin deposits are enforceable. It is one of the roles of market infrastructures to ensure such contractibility and enforceability.

We will consider two types of margins. An initial margin is a requirement to deposit cash at $t = 0$ when the protection buyer and seller enter a risk-sharing contract. A variation margin is a requirement to deposit cash at $t = 0.5$ after advance information about the risk θ is observed.

Contract: The contract specifies a transfer τ at time 2 between the protection seller and the protection buyer.[10] When $\tau > 0$ the protection seller pays the protection buyer and vice versa when $\tau < 0$. The transfer τ can be conditional on all observable information: the realization of the risk $\tilde{\theta}$, the return on the seller's assets $\tilde{R}$ and the advance signal $\tilde{s}$.

[10] In our simple framework, allowing for an upfront payment by the protection buyer at $t = 0$ would not change the analysis.

Hence, transfers are denoted by $\tau(\tilde{\theta}, \tilde{s}, \tilde{R})$. The contract also specifies margin requirements. Transfers must be consistent with the limited liability of the protection seller, so that $\alpha A + (1-\alpha)A\tilde{R} \geq \tau(\tilde{\theta}, \tilde{s}, \tilde{R})$. We assume $A \geq \pi \Delta\theta$, where $\Delta\theta \equiv \bar{\theta} - \underline{\theta}$. As we will show below, this implies that the limited liability constraint binds only if $\tilde{R} = 0$.

The sequence of events is summarized in Figure 1.

Insert Figure 1 here

3 First-best: observable effort

In this section we consider the case in which the protection buyer can observe the seller's risk-prevention effort so that there is no moral hazard and the first-best is achieved. While implausible, this case offers a benchmark against which we will identify the inefficiencies generated by moral hazard.

In the first-best, efficiency requires that the protection seller exerts effort and that margins are not used. Their benefit is to ring-fence assets from the seller's moral hazard problem, which is absent in the first-best. Since the return $\tilde{R}$ is always equal to R, we don't need to write $\tilde{R}$ among the variables upon which τ is contingent.

The protection buyer solves

$$\max_{\tau} \pi\lambda u(\bar{\theta} + \tau(\bar{\theta}, \bar{s})) + (1-\pi)(1-\lambda)u(\underline{\theta} + \tau(\underline{\theta}, \bar{s})) + \pi(1-\lambda)u(\bar{\theta} + \tau(\bar{\theta}, \underline{s})) + (1-\pi)\lambda u(\underline{\theta} + \tau(\underline{\theta}, \underline{s})) \quad (1)$$

subject to the protection seller's participation constraint

$$\pi\lambda[AR - \tau(\bar{\theta}, \bar{s})] + \pi(1-\lambda)[AR - \tau(\bar{\theta}, \underline{s})] + (1-\pi)\lambda[AR - \tau(\underline{\theta}, \underline{s})] + (1-\pi)(1-\lambda)[AR - \tau(\underline{\theta}, \bar{s})] \geq AR$$

The expression on the right-hand side of the participation constraint is the protection seller's payoff if she does not enter the transaction, in which case she exerts effort. The participation constraint simplifies to

$$0 \geq E[\tau(\tilde{\theta}, \tilde{s})] \quad (2)$$

Condition (2) states that the protection seller agrees to the contract as long as the average payment to the buyer is non-positive. Proposition 1 states the first-best outcome.

Proposition 1 (First-best contract) *When effort is observable, the optimal contract entails effort, provides full insurance and is actuarially fair. Margins are not used. The transfers are given by*

$$\begin{aligned}\tau^{FB}(\bar{\theta}, \bar{s}) &= \tau^{FB}(\bar{\theta}, \underline{s}) = -(1-\pi)\Delta\theta = E[\tilde{\theta}] - \bar{\theta} < 0 \\ \tau^{FB}(\underline{\theta}, \bar{s}) &= \tau^{FB}(\underline{\theta}, \underline{s}) = \pi\Delta\theta = E[\tilde{\theta}] - \underline{\theta} > 0\end{aligned}$$

In the first-best contract, there is no counterparty risk and the consumption of the protection buyer is equalized across states. The contract does not react to the signal. Expected transfers are zero and there are no rents to the protection seller. The payments are proportional to the riskiness of the position, measured by $\Delta\theta$ and under our assumption $A > \pi\Delta\theta$, the limited liability constraint does not bind.

4 Unobservable effort, no margins

We hereafter assume that the seller's risk-prevention effort is not observable. In this section we characterize the optimal contract assuming that margins are not used, i.e., $\alpha = 0$. This provides a useful benchmark against which we assess the effect of margins in section 5.

4.1 Effort after both signals

We first consider a contract that induces effort after both a good and a bad signal. On the equilibrium path $\tilde{R} = R$, so that transfers only need to be contingent on the risk $\tilde{\theta}$ and the signal $\tilde{s}$. The protection buyer solves (1) subject to (2) as well as now the seller's incentive compatibility constraints. Since the signal about the risk $\tilde{\theta}$ is observed before the effort decision is made, the incentive constraints are conditional on the realization of the signal.

In case of a good signal, $\tilde{s} = \bar{s}$, the incentive-compatibility constraint is given by

$$\bar{\pi}[AR - \tau(\bar{\theta}, \bar{s})] + (1-\bar{\pi})[AR - \tau(\underline{\theta}, \bar{s})] \geq$$
$$\bar{\pi}[p(AR - \tau(\bar{\theta}, \bar{s}))] + (1-\bar{\pi})[p(AR - \tau(\underline{\theta}, \bar{s}))] + AB$$

The expression on the right-hand side is the protection seller's (out-of-equilibrium) expected payoff if she does not exert effort. With probability $1-p$, her assets return zero and she cannot make any positive payment. The protection buyer, in turn, has no interest in making a payment to the protection seller when $\tilde{R} = 0$, since it would only make it more difficult to satisfy the incentive constraint. The incentive-compatibility constraint after a bad signal, $\tilde{s} = \underline{s}$, is derived analogously. Simplifying the incentive constraints we get:

$$A\mathcal{P} \geq \bar{\pi}\tau(\bar{\theta}, \bar{s}) + (1 - \bar{\pi})\tau(\underline{\theta}, \bar{s}) \text{ and } A\mathcal{P} \geq \underline{\pi}\tau(\bar{\theta}, \underline{s}) + (1 - \underline{\pi})\tau(\underline{\theta}, \underline{s}),$$

where

$$\mathcal{P} \equiv R - \frac{B}{1-p}. \tag{3}$$

Following Tirole (2005), we refer to $\mathcal{P}$ as the "pledgeable income" of the protection seller, i.e., the share of the return per unit of assets that can be pledged to an outside investor without jeopardizing the incentives of the agent managing the assets. Note that $\mathcal{P} > 0$ under our assumption that effort is efficient, i.e., $R > pR + B$. Denoting:

$$\bar{\tau} \equiv E(\tau|\bar{s}) \tag{4}$$

$$\underline{\tau} \equiv E(\tau|\underline{s}), \tag{5}$$

the incentive constraints become

$$A\mathcal{P} \geq \bar{\tau} \tag{6}$$

$$A\mathcal{P} \geq \underline{\tau} \tag{7}$$

and the participation constraint (2) becomes

$$0 \geq \text{prob}[\bar{s}]\bar{\tau} + \text{prob}[\underline{s}]\underline{\tau} \tag{8}$$

For sufficiently high levels of $\mathcal{P}$, the incentive-compatibility constraints are not binding at the first-best allocation. This leads to the following lemma.

Lemma 1 *When effort is not observable, the first-best can be achieved if and only if the pledgeable income is high enough, in the sense that*

$$A\mathcal{P} \geq (\pi - \underline{\pi})\Delta\theta = E[\tilde{\theta}] - E[\tilde{\theta}|\underline{s}].$$

The threshold level of pledgeable income beyond which the first-best is attainable is increasing in the riskiness of the position $\Delta\theta$ and the informativeness of the signal λ. Thus, Lemma 1 yields the following corollary.

Corollary 1 *When the signal is uninformative, $\lambda = \frac{1}{2}$, the first-best is always reached since $A\mathcal{P} > (\pi - \underline{\pi})\Delta\theta = 0$.*

In what follows, we focus on the case in which the first-best is not attainable and, moreover, the signal is sufficiently informative. In particular, we assume that:

$$\lambda \geq \lambda^* \equiv \frac{1-\sqrt{p}}{1-p} > \frac{1}{2}. \tag{9}$$

While relatively mild,[11] this assumption simplifies the analysis by focusing on the case in which the moral hazard problem is relatively severe.

The next lemma states that the participation constraint of the protection seller binds and the contract is actuarially fair.

Lemma 2 *When the optimal contract induces the seller's effort after both signals, her participation constraint is binding,* $E(\tau) = 0$.

To ensure that the protection seller always exerts effort, both incentive-compatibility constraints (6) and (7) must hold. But the next lemma states that only the incentive constraint after a bad signal is binding.

Lemma 3 *When the optimal contract induces the seller's effort after both signals, the incentive constraint after a good signal is slack whereas the incentive constraint after a bad signal is binding.*

Ex-ante, before the signal is observed, the derivative position is neither an asset nor a liability for the protection seller. After observing a good signal about the underlying risk, the position is likely to be profitable for the seller. She is more likely to be paid by the buyer than the other way around, which strengthens the attractiveness of risk-prevention effort to stay solvent. Good news do not generate an incentive problem. Negative news, however, make it likely that the position moves against the seller. This undermines the seller's incentives to exert effort. She has to bear the full cost of effort while the benefit of staying solvent accrues in part to the protection buyer who gets paid. This is reminiscent of the debt-overhang effect (Myers, 1977).

Building on the above analysis, the following proposition characterizes the optimal contract with effort after both signals.

Proposition 2 (Optimal contract with effort) *The optimal contract that induces effort after both signals has the following characteristics:*

[11] Note that $\lambda^*(p)$ is decreasing in p with $\lambda^* \to \frac{1}{2}$ as $p \to 1$. For reasonable values of p, the threshold λ^* is close to one half. For example, $\lambda^* = 0.59$ when $p = \frac{1}{2}$.

- *Full insurance conditional on the signal: For a given realization of the signal, the consumption of the protection buyer at time 2 is independent of the realization of* θ*.*

- *Transfers:*

$$\begin{aligned}
\tau(\bar{\theta}, \bar{s}) &= -(1-\bar{\pi})\Delta\theta - \frac{prob[\underline{s}]}{prob[\bar{s}]} A\mathcal{P} < 0 \\
\tau(\underline{\theta}, \bar{s}) &= \bar{\pi}\Delta\theta - \frac{prob[\underline{s}]}{prob[\bar{s}]} A\mathcal{P} > 0 \\
\tau(\bar{\theta}, \underline{s}) &= -(1-\underline{\pi})\Delta\theta + A\mathcal{P} < 0 \\
\tau(\underline{\theta}, \underline{s}) &= \underline{\pi}\Delta\theta + A\mathcal{P} > 0
\end{aligned}$$

The key difference to the first-best contract is that the transfers now depend on the signal. To preserve the seller's incentives to exert effort, the buyer must reduce the amount of insurance after a bad signal, $\tau(\underline{\theta}, \underline{s}) < \tau(\underline{\theta}, \bar{s})$, and thus accept incomplete risk-sharing. Hence, the protection buyer bears signal risk. Correspondingly, the protection seller must be left with some rent after a bad signal in order to exert effort. The protection buyer "reclaims" this rent after a good signal, $\tau(\bar{\theta}, \underline{s}) < \tau(\bar{\theta}, \bar{s})$, so that the expected rent to the seller is zero.

Conditional on the signal, the optimal contract provides full insurance against the underlying risk $\tilde{\theta}$:

$$\tau(\underline{\theta}, \bar{s}) - \tau(\bar{\theta}, \bar{s}) = \tau(\underline{\theta}, \underline{s}) - \tau(\bar{\theta}, \underline{s}) = \Delta\theta > 0 \tag{10}$$

Since there is full insurance conditional on the signal, we can rewrite the objective of the risk-averse protection buyer (1) as

$$\text{prob}[\bar{s}]u(E[\theta|\bar{s}] + \bar{\tau}) + \text{prob}[\underline{s}]u(E[\theta|\underline{s}] + \underline{\tau}) \tag{11}$$

where $\bar{\tau}$ and $\underline{\tau}$ are as defined in (4) and (5).

Figure 2 illustrates our results so far in the contract space $(\underline{\tau}, \bar{\tau})$. Since the incentive constraint after bad news binds, $\underline{\tau} = A\mathcal{P} > 0$, and at the same time $E(\tau) = 0$, we have $\underline{\tau} < 0$. Hence, the relevant part of the contract space is when $\underline{\tau} \geq 0$ (x-axis) and $\underline{\tau} \leq 0$ (y-axis). After a bad signal the protection seller is more likely to pay the protection buyer than vice versa. The opposite holds after a good signal.

Insert Figure 2 here

The participation constraint of the protection seller (8) is a line through the origin with slope $-\frac{\text{prob}[\underline{s}]}{\text{prob}[\bar{s}]}$. The protection seller agrees to any contract that lies on or below this line. Contracts that lie on the line are actuarially fair since expected transfers are zero. The slope gives the "relative price" at which the risk-neutral protection seller is willing to exchange expected transfers after a good and a bad signal.

The indifference curves corresponding to (11) are decreasing, convex curves in the contract space $(\underline{\tau}, \bar{\tau})$.[12] The utility of the protection buyer increases as he moves to the north-east in the figure.

The first-best allocation is given by point A where the indifference curve of the protection buyer is tangent to the participation constraint of the protection seller. Point B illustrates the optimal contract with unobservable effort. The vertical line that intersects the x-axis at $\underline{\tau} = A\mathcal{P}$ represents the incentive constraint after a bad signal. The protection seller only exerts effort after a bad signal if the contract lies on or to the left of the line. The figure is drawn for $A\mathcal{P} < E[\tilde{\theta}] - E[\tilde{\theta}|\underline{s}]$ so that the first-best allocation is not attainable when effort is not observable (Lemma 1). The contract achieving the highest utility for the protection buyer lies at the intersection of the incentive and the participation constraint. He is worse off than with the first-best allocation. The indifference curve passing through B lies strictly below the one passing through A.

4.2 No effort after a bad signal (risk-taking)

The protection buyer may find the reduced risk-sharing in the contract with effort after bad news too costly. He may instead choose to accept shirking on risk-prevention effort (risk-taking) by the protection seller in exchange for a better sharing of the risk associated with $\tilde{\theta}$. In this subsection, we characterize the optimal contract with risk-taking by the seller after a bad signal.

As before, the protection seller's incentives to exert effort are intact after good news so that $\tilde{R} = R$. After bad news, the seller now does not exert effort so that $\tilde{R} = R$ with probability p and $\tilde{R} = 0$ with probability $1 - p$. Hence, the contractual transfer τ must now

[12]The slope of an indifference curve is given by $\frac{d\bar{\tau}}{d\underline{\tau}} = -\frac{\text{prob}[\underline{s}]\underline{u}'}{\text{prob}[\bar{s}]\bar{u}'} < 0$, where $\underline{u}' \equiv u'(E[\theta|\underline{s}] + \underline{\tau})$ and $\bar{u}' \equiv u'(E[\theta|\bar{s}] + \bar{\tau})$. The change in the slope is $\frac{d^2\bar{\tau}}{d\underline{\tau}^2} = -\frac{\text{prob}[\underline{s}]\underline{u}''\bar{u}'}{(\text{prob}[\bar{s}]\bar{u}')^2} > 0$.

be contingent on the realization of $\tilde{R}$. The objective of the protection buyer is given by

$$\max_{\tau} \pi\lambda u(\bar{\theta}+\tau(\bar{\theta},\bar{s},R)) + (1-\pi)(1-\lambda)u(\underline{\theta}+\tau(\underline{\theta},\bar{s},R)) \\ + \pi(1-\lambda)[pu(\bar{\theta}+\tau(\bar{\theta},\underline{s},R)) + (1-p)u(\bar{\theta}+\tau(\bar{\theta},\underline{s},0))] \\ + (1-\pi)\lambda[pu(\underline{\theta}+\tau(\underline{\theta},\underline{s},R)) + (1-p)u(\underline{\theta})] \tag{12}$$

With probability $1-p$ the seller's assets return zero and she cannot make any transfers to the protection buyer. It may, however, be optimal for the buyer to make a transfer to the seller when she defaults but the good state $\bar{\theta}$ is realized, i.e., it may be optimal to set $\tau(\bar{\theta},\underline{s},0) < 0$. In contrast, the transfer when the seller defaults and the bad state $\underline{\theta}$ is realized is optimally set to zero, $\tau(\underline{\theta},\underline{s},0) = 0$. Indeed, the protection buyer would like to receive an insurance payment in the bad state $\underline{\theta}$, but when $\tilde{R} = 0$ the protection seller is unable to make any payment.

The seller's incentive constraint after good news is, as before,

$$A\mathcal{P} \geq \bar{\pi}\tau(\bar{\theta},\bar{s},R) + (1-\bar{\pi})\tau(\underline{\theta},\bar{s},R), \tag{13}$$

whereas after bad news, the seller must prefer not to exert effort

$$\underline{\pi}[AR - \tau(\bar{\theta},\underline{s})] + (1-\underline{\pi})[AR - \tau(\underline{\theta},\underline{s})] \leq \\ \underline{\pi}[p(AR - \tau(\bar{\theta},\bar{s})) - (1-p)\,\tau(\bar{\theta},\underline{s},0)] + (1-\underline{\pi})[p(AR - \tau(\underline{\theta},\bar{s}))] + AB,$$

or, equivalently,

$$A\mathcal{P} \leq \underline{\pi}\tau(\bar{\theta},\underline{s},R) + (1-\underline{\pi})\tau(\underline{\theta},\underline{s},R) - \underline{\pi}\tau(\bar{\theta},\underline{s},0). \tag{14}$$

The seller's participation constraint with risk-taking is

$$-\,\mathrm{prob}[\underline{s}](1-p)A\mathcal{P} \geq \mathrm{prob}[\bar{s}]\left[\bar{\pi}\tau(\bar{\theta},\bar{s},R) + (1-\bar{\pi})\tau(\underline{\theta},\bar{s},R)\right] + \\ \mathrm{prob}[\underline{s}]p\left[\underline{\pi}\tau(\bar{\theta},\underline{s},R) + (1-\underline{\pi})\tau(\underline{\theta},\underline{s},R)\right] + \mathrm{prob}[\underline{s}]\,(1-p)\,\underline{\pi}\tau(\bar{\theta},\underline{s},0) \tag{15}$$

The expected transfer from the seller to the buyer (right-hand side) is negative. If the seller enters the position, she must be compensated for the potential efficiency loss due to the lack of effort after bad news (left-hand side). Thus, the contract with no effort after bad news is actuarially unfair. The higher the pledgeable income, the greater is the efficiency loss generated by risk-taking after bad news and the more actuarially unfair is the contract. The participation constraint (15) implies that apparently expensive derivative contracts sold by well established institutions (high $\mathcal{P}$) can be an indication of future risk-taking.

Building on the above analysis, we obtain our next proposition.

Proposition 3 (Optimal contract with risk-taking) *If risk-taking (no effort) is preferred to effort after bad news, then the optimal contract provides full insurance except when the seller defaults in the* $\underline{\theta}$ *state. The transfers are given by* $\tau(\underline{\theta}, \underline{s}, 0) = 0$ *and*

$$\begin{aligned} \tau(\bar{\theta}, \bar{s}, R) &= \tau(\bar{\theta}, \underline{s}, R) = \tau(\bar{\theta}, \underline{s}, 0) = \frac{\pi \Delta\theta - prob[\underline{s}]\,(1-p)\,A\mathcal{P}}{1 - prob[\underline{s}]\,(1-\underline{\pi})\,(1-p)} - \Delta\theta < 0 \\ \tau(\underline{\theta}, \bar{s}, R) &= \tau(\underline{\theta}, \underline{s}, R) = \frac{\pi \Delta\theta - prob[\underline{s}]\,(1-p)\,A\mathcal{P}}{1 - prob[\underline{s}]\,(1-\underline{\pi})\,(1-p)} > 0 \end{aligned}$$

While the contract is actuarially unfair, there are no rents to the protection seller since the participation constraint is binding. The seller pays the buyer in the bad state $\tilde{\theta} = \underline{\theta}$ if she does not default and vice versa in the good state $\tilde{\theta} = \bar{\theta}$: $\tau(\underline{\theta}, \tilde{s}, R) > 0 > \tau(\bar{\theta}, \tilde{s}, \tilde{R})$. In contrast to the contract when the seller does exert effort after bad news, the contract without such effort does not react to the signal, i.e., $\tau(\tilde{\theta}, \bar{s}) = \tau(\tilde{\theta}, \underline{s})$. Except when the protection seller defaults and the bad state $\underline{\theta}$ occurs, the consumption of the buyer is equalized across states (as in the first-best contract). But when the protection seller defaults and $\underline{\theta}$ occurs, the buyer cannot receive any insurance payment and is therefore exposed to counterparty risk.

4.3 Risk-sharing and risk-taking

The contract under which the protection seller exerts effort after both signals entails limited risk-sharing for the buyer but has no risk-taking by the seller (Section 4.1), while the contract with no effort after a bad signal entails full risk-sharing for the protection buyer unless the seller defaults due to risk-taking (Section 4.2). The next proposition characterizes the privately optimal choice between the two contracts.

Proposition 4 (Endogenous counterparty risk) *There exists a threshold level of per-unit pledgeable income* $\hat{\mathcal{P}}$ *such that the contract with risk-prevention effort after a bad signal is optimal if and only if* $\mathcal{P} \geq \hat{\mathcal{P}}$*. If the probability of default* $1-p$ *is sufficiently small, then* $\hat{\mathcal{P}} > 0$*.*

The key factor in the choice is whether signal risk or counterparty risk is more costly for the protection buyer. For low levels of pledgeable income, the moral hazard is severe. Maintaining the seller's incentives after a bad signal requires a considerable reduction in insurance. The buyer then has to bear a lot of signal risk. If at the same time default is unlikely (p is high), then it is optimal to allow the seller to shirk on the risk-prevention effort at the cost of counterparty risk. The proposition also implies that risk-taking by the protection seller is more likely when the return on her asset (R) is low.

4.4 Risk-taking or risk-shifting?

So far, we modeled moral hazard in terms of an effort to increase returns in the sense of first-order stochastic dominance. In this subsection, we show that the problem is equivalent when we consider an unobservable action that worsens returns in the sense of second-order stochastic dominance. Such an alternative formulation of moral hazard is in line with the risk-shifting problem identified by Jensen and Meckling (1976).

Assume the per-unit return on the protection seller's balance sheet, $\tilde{R}$, can be high (H), medium (M), or low (L), with $H > M > L$. For simplicity, normalize L to 0. As in Biais and Casamatta (1999), the protection seller makes an unobservable choice about the probability distribution over H, L and M. She can choose a relatively safe distribution for which the return is H with probability $1-\mu$ and M with probability μ. Denote the expected return in this case by $E[R]$. Alternatively, she can choose a riskier distribution, i.e., engage in risk-shifting, where the return is H with probability $1-\mu+\alpha$, M with probability $\mu-(\alpha+\beta)$, and L with probability β. Denote the expected return in this case by $\hat{E}[R]$. We assume that $E[R] > \hat{E}[R]$, i.e., that the expected return is lower with risk-shifting than without. Unlike in the moral hazard problem analyzed previously, there is no private benefit. Yet, the protection seller can be tempted to engage in risk-shifting.

As before, the hedging contract between the protection seller and the protection buyer specifies the transfers as a function of the realizations of $\tilde{s}$, $\tilde{R}$ and $\tilde{\theta}$. If the contract entails no risk-shifting by the protection seller, her participation constraint is given by

$$AE[R] - [\text{prob}[\bar{s}]\bar{\tau} + \text{prob}[\underline{s}]\underline{\tau}] \geq AE[R],$$

where $\bar{\tau}$ and $\underline{\tau}$ are as defined in (4) and (5). Equivalently, the participation constraint can be written as $\text{prob}[\bar{s}]\bar{\tau}+\text{prob}[\underline{s}]\underline{\tau} \leq 0$, which is identical to (8).

The incentive constraints of the protection seller now ensure no risk-shifting. Assuming that the return M is large enough for the protection seller to not default when M occurs, we have the following two incentive constraints:

$$\begin{aligned}(1-\mu)(AH-\bar{\tau})+\mu(AM-\bar{\tau}) &\geq (1-\mu+\alpha)(AH-\bar{\tau})+(\mu-(\alpha+\beta))(AM-\bar{\tau}),\\ (1-\mu)(AH-\underline{\tau})+\mu(AM-\underline{\tau}) &\geq (1-\mu+\alpha)(AH-\underline{\tau})+(\mu-(\alpha+\beta))(AM-\underline{\tau}).\end{aligned}$$

The incentive constraints simplify to

$$A\bar{\mathcal{P}} \geq \bar{\tau} \tag{16}$$

$$A\bar{\mathcal{P}} \geq \underline{\tau} \tag{17}$$

where

$$\bar{\mathcal{P}} \equiv -\left[\frac{\alpha}{\beta}(H-M)-M\right]. \tag{18}$$

The pledgeable return $\bar{\mathcal{P}}$ of risk-shifting is the counterpart of $\mathcal{P}$ in the case of risk-prevention effort. Both are given by the difference in expected returns under the efficient and the inefficient action, divided by the probability of default under the inefficient action (β here and $(1-p)$ before). The incentive constraints (16) and (17) are similar to (6) and (7). The objective of the protection buyer (1) is unchanged since the limited liability constraint of the protection seller does not bind when she does not engage in risk-shifting and since optimal transfers do not depend on whether the return H or M realizes. Hence, the optimal contract without risk-shifting is the same as the one characterized in Subsection 4.1, up to a re-definition of the pledgeable income from $\mathcal{P}$ to $\bar{\mathcal{P}}$. All the qualitative effects are the same, which implies that our economic message is robust to the specification of the moral hazard, whether in terms of first- or second-order stochastic dominance.

Consider now the contract with risk-shifting by the protection seller after bad news. The objective of the protection buyer is given by

$$\begin{aligned}\max_{\tau}\ & \pi\lambda u(\bar{\theta}+\tau(\bar{\theta},\bar{s},H \text{ or } M)) + (1-\pi)(1-\lambda)u(\underline{\theta}+\tau(\underline{\theta},\bar{s},H \text{ or } M)) \\ & + \pi(1-\lambda)[(1-\beta)\,u(\bar{\theta}+\tau(\bar{\theta},\underline{s},H \text{ or } M)) + \beta u(\bar{\theta}+\tau(\bar{\theta},\underline{s},0))] \\ & + (1-\pi)\lambda[(1-\beta)\,u(\underline{\theta}+\tau(\underline{\theta},\underline{s},H \text{ or } M)) + \beta u(\underline{\theta})],\end{aligned} \tag{19}$$

The objective is similar to (12), with p replaced by $(1-\beta)$. The participation constraint of the protection seller under risk-shifting is

$$\begin{aligned}&-\text{prob}[\underline{s}]\beta A\bar{\mathcal{P}} \geq \text{prob}[\bar{s}]\left[\bar{\pi}\tau(\bar{\theta},\bar{s},\tilde{R}) + (1-\bar{\pi})\tau(\underline{\theta},\bar{s},\tilde{R})\right] + \\ &\text{prob}[\underline{s}]\,(1-\beta)\left[\underline{\pi}\tau(\bar{\theta},\underline{s},\tilde{R}\geq M) + (1-\underline{\pi})\tau(\underline{\theta},\underline{s},\tilde{R}\geq M)\right] + \text{prob}[\underline{s}]\beta\underline{\pi}\tau(\bar{\theta},\underline{s},0),\end{aligned} \tag{20}$$

which is similar to the one without risk-prevention effort, (15). As for the incentive constraints, we know from Section 4.2 that they do not bind when the contract without risk-prevention effort is optimal. Hence, there is no need to consider them explicitly. We conclude that the problem with risk-shifting after bad news is isomorphic to the problem without effort, or risk-taking, after bad news.

5 Margins

We now turn to the case in which margins can be used. In the case of the contract with effort after bad news and limited risk-sharing for the protection buyer, we show how margins improve incentives. In the case of the contract without effort after bad news and counterparty risk, we examine the role of margins in providing insurance against the protection seller's default.

5.1 Margins under effort

When the protection seller exerts effort, she always obtains the return R and can always pay the transfer τ. Margins need not be used to pay the protection buyer at $t = 2$. The objective of the protection buyer is unchanged and is given by (1) or, equivalently, by (11). Furthermore, the incentive constraint of the protection seller does not bind after a good signal. Calling a margin would only incur the inefficiency of liquidating assets without any benefit for incentives. Hence, the protection buyer will not make a margin call after a good signal. Hence, it is also not optimal to make a margin call before the signal realizes. But to maintain the protection seller's incentives after a bad signal, the protection buyer may request the seller to liquidate a fraction α of her assets and deposit the resulting cash on a margin account. Calling the margin conditional on the bad signal means it is a variation margin.

The seller's participation constraint now is

$$\text{prob}[\bar{s}]AR + \text{prob}[\underline{s}]\left[\alpha A + (1-\alpha)AR\right] - E[\tau] \geq AR$$

or, equivalently,

$$E[\tau] \leq -\alpha A (R-1)\,\text{prob}[\underline{s}] \tag{21}$$

The expression on the right-hand side is negative and represents the protection seller's expected opportunity cost of liquidating assets after a bad signal. She forgoes the net return of assets over cash, $R-1$. To offset the opportunity cost of the variation margin and make the seller willing to enter the contract, the expected transfer to the protection buyer must be negative, i.e., the contract is actuarially unfair.

The incentive-compatibility constraint after a bad signal now is

$$\alpha A + (1-\alpha)AR - \underline{\tau} \geq p\left[\alpha A + (1-\alpha)AR - \underline{\tau}\right] + (1-\alpha)AB$$

The expression on the right-hand side is the protection seller's (out-of-equilibrium) expected payoff if she does not exert effort. She earns the private benefit B only on the assets she has not had to liquidate. There is no private benefit associated with cash deposited on a margin account. Higher margins thus reduce the private benefit of shirking on risk-prevention effort. When the seller's assets return zero (on the out-of-equilibrium path), the cash deposited is used to pay the buyer.[13] We can rewrite the incentive constraint as

$$\alpha A + (1-\alpha) A\mathcal{P} \geq \underline{\tau} \tag{22}$$

where $\mathcal{P}$ denotes, as before, the pledgeable income per unit of assets. Relying on standard arguments, one can show that (21) and (22) must bind in the optimal contract, as stated in the following lemma.

Lemma 4 *In the optimal contract with margins and effort, the participation constraint of the protection seller, as well as the incentive-compatibility constraint after bad news, are binding.*

As can be seen in (22), margins tighten the incentive constraint when $\mathcal{P} \geq 1$. Since they also tighten the participation constraint, they are then suboptimal. This yields the following lemma:

Lemma 5 *If $\mathcal{P} \geq 1$, margins are not used.*

We therefore turn to the case when margins can be optimal, $\mathcal{P} < 1$. Using Lemma 4 we obtain the expected transfers conditional on the signal:

$$\underline{\tau}(\alpha) = \alpha A + (1-\alpha) A\mathcal{P}, \tag{23}$$

and

$$\bar{\tau}(\alpha) = -\frac{\text{prob}[\underline{s}]}{\text{prob}[\bar{s}]}\left[\alpha A R + (1-\alpha) A\mathcal{P}\right]. \tag{24}$$

Equation (23) implies that $\frac{\partial \underline{\tau}}{\partial \alpha} > 0$. By reducing the amount of assets subject to moral hazard, margins relax the incentive constraint and increase the expected transfer to the protection buyer after a bad signal. Equation (24) implies that $\frac{\partial \bar{\tau}}{\partial \alpha} < 0$. Margins impose

[13] In the contract with effort, it is optimal for the buyer to seize the margin whenever the seller is in default both in the $\underline{\theta}$ and the $\bar{\theta}$ state. This is because returning the margin would tighten the incentive constraint.

an opportunity cost on the protection seller who requests a higher expected transfer after a good signal to offset this cost.

To quantify the effect of a margin α, it is useful to introduce the function φ, defined as follows

$$\varphi(\alpha) \equiv \frac{u'(E[\theta|\underline{s}] + \underline{\tau}(\alpha))}{u'(E[\theta|\bar{s}] + \bar{\tau}(\alpha))}.$$

Given (23) and (24), φ is a known function of exogenous variables and α. It is the ratio of the marginal utility of the protection buyer after a bad and a good signal. In the first-best, there is full insurance and φ is equal to 1. With moral hazard, insurance can be imperfect and φ can be greater than one. Since $\frac{\partial \bar{\tau}}{\partial \alpha} < 0$ and $\frac{\partial \underline{\tau}}{\partial \alpha} > 0$, φ is decreasing. Higher margins reduce φ, moving the expected transfers closer to full insurance. The margin improves risk-sharing even though it actually is never transferred from the protection seller to the buyer.

The following proposition characterizes the optimal contract with margins and effort when $\mathcal{P} < 1$.

Proposition 5 (Optimal margins with effort) *Consider the case $\mathcal{P} < 1$. If $\varphi(0) < 1 + \frac{R-1}{1-\mathcal{P}}$, then it is optimal not to use margins. If $\varphi(1) > 1 + \frac{R-1}{1-\mathcal{P}}$, then the optimal margin is $\alpha^* = 1$. Otherwise, there exists an optimal margin, $\alpha^* \in (0,1)$ such that*

$$\varphi(\alpha^*) = 1 + \frac{R-1}{1-\mathcal{P}}. \tag{25}$$

The optimal margin balances inefficient liquidation and enhanced insurance. The right-hand side of (25) gives the rate at which this tradeoff occurs. The numerator of the fraction, $R-1$, is the opportunity cost of liquidation. The denominator measures the severity of the incentive problem that limits insurance. Margins can relax the incentive problem (22) only for $\mathcal{P} < 1$, and they are particularly beneficial when the pledgeable income, $\mathcal{P}$, is low.

Figure 3 illustrates the analysis of margins when the optimal contract entails effort by the protection seller after a bad signal. The margin affects the participation constraint and the incentive constraint after a bad signal but leaves the objective function of the protection buyer unchanged. The straight line from point B to point D illustrates how the margin changes the set of feasible contracts. The line is the parametric plot of the binding participation constraint (21) and incentive constraint after bad news (22) as α varies from 0 to 1. The point B represents the optimal contract with effort and no margin (see Section 4.1). The optimal margin α^* is given by the point of tangency of the protection buyer's indifference curve to the line BD (point E).[14]

Insert Figure 3 here

5.2 Margins under risk-taking after a bad signal

When the protection seller does not exert effort after a bad signal and thus engages in risk-taking, the objective of the protection buyer is

$$\begin{aligned}\max_{\alpha,\tau}\ & \pi\lambda u(\bar{\theta}+\tau(\bar{\theta},\bar{s},R))+(1-\pi)(1-\lambda)u(\underline{\theta}+\tau(\underline{\theta},\bar{s},R)) \\ & +\pi(1-\lambda)[pu(\bar{\theta}+\tau(\bar{\theta},\underline{s},R))+(1-p)u(\bar{\theta}+\tau(\bar{\theta},\underline{s},0))] \\ & +(1-\pi)\lambda[pu(\underline{\theta}+\tau(\underline{\theta},\underline{s},R))+(1-p)u(\underline{\theta}+\tau(\underline{\theta},\underline{s},0))]\end{aligned} \tag{26}$$

where $\tau(\tilde{\theta},\underline{s},0)\leq\alpha A$. The margin allows to pay the protection buyer $\tau(\tilde{\theta},\underline{s},0)$ when the protection seller defaults up to the amount that was deposited on the margin account.

The participation constraint of the protection seller is now given by

$$\begin{aligned}-\text{prob}[\underline{s}]\left[\alpha A\left(R-1\right)+(1-p)(1-\alpha)A\mathcal{P}\right] &\geq \text{prob}[\bar{s}]\left[\bar{\pi}\tau(\bar{\theta},\bar{s},R)+(1-\bar{\pi})\tau(\underline{\theta},\bar{s},R)\right] \\ +\text{prob}[\underline{s}]p\left[\underline{\pi}\tau(\bar{\theta},\underline{s},R)+(1-\underline{\pi})\tau(\underline{\theta},\underline{s},R)\right] &+\text{prob}[\underline{s}]\left(1-p\right)\left(\underline{\pi}\tau(\bar{\theta},\underline{s},0)+(1-\underline{\pi})\tau(\underline{\theta},\underline{s},0)\right)\end{aligned} \tag{27}$$

The left-hand side is the sum of the protection seller's opportunity cost of depositing a margin and the loss of pledgeable income when she defaults. The right-hand side is the expected transfer from the protection seller to the protection buyer. The incentive constraint after bad news is

$$\left(1-\alpha\right)A\mathcal{P}\leq\underline{\pi}\tau(\bar{\theta},\underline{s},R)+(1-\underline{\pi})\tau(\underline{\theta},\underline{s},R)-\underline{\pi}\tau(\bar{\theta},\underline{s},0)-(1-\underline{\pi})\tau(\underline{\theta},\underline{s},0). \tag{28}$$

As before, the protection seller does not obtain rents:

Lemma 6 *In the optimal contract with margins and risk-taking after bad news, the participation constraint of the protection seller binds.*

The next lemma narrows down the parameter space for which risk-taking after bad news can be optimal:

[14]At the point of tangency, we have $-\frac{\text{prob}[\underline{s}]}{\text{prob}[\bar{s}]}\varphi\left(\alpha^*\right)=-\frac{\text{prob}[\underline{s}]}{\text{prob}[\bar{s}]}-\frac{(R-1)\text{prob}[\underline{s}]}{(1-\mathcal{P})\text{prob}[\bar{s}]}$. Multiplying both sides of the equality with $-\frac{\text{prob}[\bar{s}]}{\text{prob}[\underline{s}]}$ recovers condition (25).

Lemma 7 *If $pR + B < 1$, then $\alpha^* = 1$ in the contract with risk-taking after bad news. This contract is, however, weakly dominated by the one with effort bad news.*

Without effort after a bad signal, the expected per-unit return on the seller's balance sheet including the private benefit is $pR + B$. If $pR + B < 1$, this is lower than the return on cash. Hence, it is more profitable to deposit all of the protection seller's assets in the margin account, $\alpha = 1$, where they earn a greater return and are ring-fenced from moral hazard. But the protection buyer can do at least as well by requesting effort after bad news since, there too, $\alpha = 1$ can be selected. It follows that the contract with margins and no effort after bad news can only be strictly optimal if $pR + B \geq 1$.

The next lemma characterizes the optimal transfers without effort after bad news when margins can be used.

Lemma 8 *If risk-taking is preferred to effort after bad news, then the incentive compatibility condition after bad news is slack. The optimal contract provides full insurance except when the seller defaults in the state $\underline{\theta}$. The transfers are given by $\tau(\underline{\theta}, \bar{s}, R) = \tau(\underline{\theta}, \underline{s}, R)$ and $\tau(\bar{\theta}, \bar{s}, R) = \tau(\bar{\theta}, \underline{s}, R) = \tau(\bar{\theta}, \underline{s}, 0) = \tau(\underline{\theta}, \underline{s}, R) - \Delta\theta$ where*

$$\tau(\underline{\theta}, \underline{s}, R) = \frac{\pi\Delta\theta - prob[\underline{s}]\,(1-p)\,A\mathcal{P}}{1 - prob[\underline{s}]\,(1-\underline{\pi})\,(1-p)} - \alpha A \frac{prob[\underline{s}]\,[pR + B - 1 + (1-\underline{\pi})\,(1-p)]}{1 - prob[\underline{s}]\,(1-\underline{\pi})\,(1-p)} \qquad (29)$$

and

$$\tau(\underline{\theta}, \underline{s}, 0) = \alpha A.$$

When risk-taking by the protection seller is preferred after bad news, then the incentive compatibility constraints are not binding and the insurance of the protection buyer is only limited by the default of the protection seller. The usefulness of margins is that they insure the protection buyer against such default by paying $\tau(\underline{\theta}, \underline{s}, 0)$ to the protection buyer. Since margins entail an opportunity cost of liquidation, there is no point in depositing more in the margin account than what is needed to insure against the counterparty risk, $\tau(\underline{\theta}, \underline{s}, 0)$.

Similar to before, define

$$\phi(\alpha) \equiv \frac{u'(\underline{\theta} + \tau(\underline{\theta}, \underline{s}, 0))}{u'(\underline{\theta} + \tau(\underline{\theta}, \underline{s}, R))},$$

which is the ratio of the marginal utility of the protection buyer in the state $\underline{\theta}$ when the protection seller defaults and when she does not default. The following proposition completes the characterization of the optimal contract with margins when the protection seller does not exert risk-prevention effort after bad news.

Proposition 6 (Optimal margins with risk-taking) *If* $\phi(0) < 1 + \frac{pR+B-1}{(1-p)(1-\underline{\pi})}$, *then it is optimal not to use margins. If* $\phi(1) > 1 + \frac{pR+B-1}{(1-p)(1-\underline{\pi})}$, *then risk-taking after a bad signal is not optimal. Otherwise, there exists an optimal margin* $\alpha^* \in (0,1)$ *such that*

$$\phi(\alpha^*) = 1 + \frac{pR + B - 1}{(1-p)(1-\underline{\pi})} \tag{30}$$

The ratio of marginal utilities ϕ is a known function of exogenous variables and the margin α. Lemma 8 implies that ϕ is decreasing in α. Margins reduce the wedge between the protection buyer's marginal utility when the seller defaults and when she does not. Hence, margins insure against counterparty risk. Similar to Proposition 5, the right-hand side of (30) gives the rate at which an optimal margin trades off inefficient liquidation and enhanced insurance, this time against counterparty risk.

When the contract entails risk-prevention effort by the seller, the margin helps to maintain her incentives after a bad signal. When the contract entails risk-taking, the margin protects the protection buyer against the default of the seller. The choice between the two contracts depends again on whether counterparty or signal risk is costlier for the protection buyer. As in Subsection 4.3, the contract with risk-taking may be chosen when pledgeable income is low and the moral hazard problem is severe.

The overall effect of margins on risk-taking, and hence counterparty risk, is ambiguous. On one hand, margins improve the protection sellers risk-prevention incentives, which makes it more likely that the protection buyer chooses the contract without counterparty risk. On the other hand, margins protect the buyer from counterparty risk, which makes it more likely that he indeed chooses the contract with such risk.

6 Multiple protection sellers

Instead of one protection seller with assets A, consider N protection sellers, indexed by $i = 1, \ldots, N$, each with $\frac{A}{N}$. Each of the N assets generates a return R per unit if the protection seller exerts risk-prevention effort. When a protection sellers does not exert effort, she exposes herself to a macro-shock that is common to all sellers. If the shock realizes (which happens with probability $1-p$), then assets return 0 for those protection sellers who did not exert effort. For simplicity, we conduct the analysis for the case in which there are no margins.

We first study the situation in which protection sellers cannot retrade or reinsure the contract. The objective of the protection buyer now is

$$\max_{\bar{\tau}_i, \underline{\tau}_i} \text{prob}[\bar{s}]u(E[\tilde{\theta}|\bar{s}] + \sum_{i=1}^{N} \bar{\tau}_i) + \text{prob}[\underline{s}]u(E[\tilde{\theta}|\underline{s}] + \sum_{i=1}^{N} \underline{\tau}_i),$$

subject to the participation and incentive constraints

$$E[\tau_i] \leq 0, \; A\mathcal{P}_i \geq \bar{\tau}_i \text{ and } A\mathcal{P}_i \geq \underline{\tau}_i, i \in \{1, ..., N\},$$

where

$$A\mathcal{P}_i = \frac{A}{N}\left(R - \frac{B}{1-p}\right).$$

The solution to this program is stated the following proposition.

Proposition 7 (Multiple sellers) *The optimal contract inducing N protection sellers to exert risk-prevention effort is given by $\bar{\tau}_i = \frac{\bar{\tau}}{N}$ and $\underline{\tau}_i = \frac{\underline{\tau}}{N}$, $i = 1, \ldots, N$, where $\bar{\tau}$ and $\underline{\tau}$ are the expected transfers conditional on the signal defined in (4) and (5), evaluated at the optimal transfers given in Proposition 2.*

Proposition 7 states that when the N protection sellers cannot retrade the contract, then the aggregate outcome for the protection buyer is the same as with only one protection seller. The linear participation and incentive constraints can be aggregated so that the optimization problem with N protection sellers of size $\frac{A}{N}$ is equivalent to the problem with one seller of size A. Thus, as in Proposition 2, the participation constraints of the protection sellers, as well as their incentive compatibility condition after a bad signal, bind.

Next, we turn to the situation in which protection sellers can retrade the contract among themselves. In the CDS market, such retrading is referred to as novation. As explained in Duffie, Li and Lubke (2010), this corresponds to the following situation: there is an original trade between parties A and B, but B wants to exit its position and pass it on to party C. After novation, there is a new counterparty relationship between A and C. As documented in Duffie et al. (2010), before 2005 the novation process could take place without the consent or even the awareness of A. But then "regulators required that dealers adopt the novation protocol, ensuring that all parties would henceforth be aware of the identities of their counterparties at all times."

For simplicity, we focus on $N = 2$. Our starting point are the contracts between the protection buyer and each protection sellers that implement the second best with effort

(Proposition 7). Now consider the possibility for protection seller 1 to acquire the contract held by seller 2 and suppose the transaction frees seller 2 from all obligations towards the protection buyer stemming from the contract.

Before $\tilde{s}$ is observed, protection seller 2 is indifferent between retrading the contract for a value of zero and keeping it since her participation constraint binds. What is protection seller 1's gain from taking over protection seller 2's contract at a price of zero? Prior to acquiring 2's contract, 1's incentive constraint after observing a bad signal was binding. Hence, increasing her position from $\frac{\underline{\tau}}{2}$ to $\underline{\tau}$ leads her to shirk on risk-prevention effort. Her expected gain is

$$\text{prob}[\bar{s}]\left(-\frac{\bar{\tau}}{2}\right)+\text{prob}[\underline{s}]\left[\frac{AB}{2}-(1-p)\frac{AR}{2}+(1-p)\left(\frac{\underline{\tau}}{2}\right)+p\left(-\frac{\underline{\tau}}{2}\right)\right].$$

The first term is the extra payment from the protection buyer that protection seller 1 expects after a good signal. The term in square brackets is the expected gain after a bad signal. The gain has four components. First, protection seller 1 obtains private benefit $\frac{AB}{2}$ by shirking on effort. Second, with probability $(1-p)$ she defaults and loses her assets. However, in case of default she no longer has to pay the protection buyer, which is the third component. Finally, with probability p she does not default and has to make payments to the protection buyer. Using $\frac{\underline{\tau}}{2}=\frac{A\mathcal{P}}{2}=\frac{A}{2}\left(R-\frac{B}{1-p}\right)$ and $E\left[\frac{\tau}{2}\right]=0$ from the binding incentive and participation constraints, seller 1's gain simplifies to

$$\text{prob}[\underline{s}]\,(1-p)\,\frac{A\mathcal{P}}{2}, \tag{31}$$

which is strictly positive. Hence, we can state our next proposition:

Proposition 8 (Retrading) *When the protection sellers can retrade the contract and transfer all the corresponding obligations, the second-best outcome with effort is not an equilibrium.*

The contract in Proposition 7 reflects the anticipation of the protection buyer that the protection sellers will not default. Hence, the buyer is willing to pay a large transfer to sellers when the good state $\bar{\theta}$ occurs expecting, in return, receiving payments from sellers when the bad state $\underline{\theta}$ occurs. But, after retrading, some protection sellers have built up positions whose embedded liability exceeds their pledgeable income. This undermines their incentives to exert effort. Such an excessive concentration of contracts, and the corresponding counterparty risk, reduce the value of the insurance payment promised to the protection buyer. The protection buyer's expected loss is the expected gain of the protection sellers.

To restore optimality, retrading must be regulated. One possibility is to require that novation be allowed only if the protection buyer agrees with it. As discussed in Duffie et al. (2010), such a novation protocol has been put forward by the International Swaps and Derivatives Association. Alternatively, *initial* margins designed to make accumulating contracts costly can restore incentives. Such margins must be deposited *before* the signal realizes (in contrast to the variation margin analyzed above). Consider again the case of two protection sellers. If both sellers retain the contract in Proposition 7, no initial margin is required. In contrast, if one of the protection sellers wants to retrade, which leads to a position incommensurate to her pledgeable income, she must put up an initial margin. If the cost of liquidating assets to comply with the initial margin (α_0) exceeds the gain from retrading (31), i.e., if

$$\text{prob}[\underline{s}]\,(1-p)\,\frac{A\mathcal{P}}{2} \leq \frac{A}{2}\,(R-1)\,\alpha_0,$$

then the second-best outcome with effort stated in Proposition 7 cannot be destabilized by retrading.

7 Conclusion

We analyze contracting between protection sellers and buyers. We show how contracts designed to engineer risk-sharing generate incentives for risk-taking. When the position of the protection seller becomes a liability for her, it undermines her incentives to exert risk-prevention effort. The failure to exert such effort may lead to the default of a protection seller. Thus, bad news about derivative positions can propagate to other lines of business of financial institutions and, when doing so, create endogenous counterparty risk for protection buyers.

When the seller's moral hazard is moderate, margins enhance the scope for risk-sharing. Initial margins discourage retrading and the accumulation of excessive derivatives positions, while variation margins discourage risk-taking for a given position. These results contrast with those of Brunnermeier and Pedersen (2009) who show that margins can be destabilizing. The contrast stems from the differences between their assumption and ours. Brunnermeier and Pedersen (2009) take margin constraints as given and, for these margins, derive equilibrium prices. Greater margins force intermediaries to sell more after bad shocks, which pushes prices down and can generate spirals. In contrast, we endogenize margins, but take as given the price at which a protection seller liquidates some of her position to deposit cash on a margin account. It would be interesting, in future research, to combine the two approaches and study how endogenous margins could be destabilizing when prices are endogenous. Destabilization could arise if the margin requirement that is privately optimal for a protection buyer and his counterparty had external effects on other investors via equilibrium prices as in Brunnermeier and Pedersen (2009). Such research would be in line with the macroprudential approach put forward by Hanson, Kashyap and Stein (2011) highlighting the general effects that arise when many financial institutions attempt to shrink their assets simultaneously.

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Appendix

Proof of Proposition 1 Let μ denote the Lagrange multiplier on the participation constraint (2). The first-order conditions are

$$\begin{aligned}
\pi\lambda u'(\bar{\theta}+\tau(\bar{\theta},\bar{s}))-\mu\pi\lambda &= 0\\
(1-\pi)(1-\lambda)u'(\underline{\theta}+\tau(\underline{\theta},\bar{s}))-\mu(1-\pi)(1-\lambda) &= 0\\
\pi(1-\lambda)u'(\bar{\theta}+\tau(\bar{\theta},\underline{s}))-\mu\pi(1-\lambda) &= 0\\
(1-\pi)\lambda u'(\underline{\theta}+\tau(\underline{\theta},\underline{s}))-\mu(1-\pi)\lambda &= 0
\end{aligned}$$

It follows that the marginal utility of the protection buyer is equalized across states and that the participation constraint is binding: $\bar{u}'(\tau(\bar{\theta},\bar{s}))=\bar{u}'(\tau(\bar{\theta},\underline{s}))=\underline{u}'(\tau(\underline{\theta},\underline{s}))=\underline{u}'(\tau(\underline{\theta},\bar{s}))=\mu>0$, where we use the shorthand $\bar{u}'(\tau(\bar{\theta},\tilde{s}))$ and $\underline{u}'(\tau(\underline{\theta},\tilde{s}))$ to denote marginal utility in state $\bar{\theta}$ and $\underline{\theta}$, respectively, conditional on the signal $\tilde{s}$. The optimal transfers are obtained by using the fact that the participation constraint is binding and that consumption is the same across all $(\tilde{\theta},\tilde{s})$ states. QED

Proof of Lemma 1 Plugging the first-best transfers from Proposition 1 into the incentive conditions (6), (7) and (8), yields $A\mathcal{P}\geq(\pi-\bar{\pi})\Delta\theta$ and $A\mathcal{P}\geq(\pi-\underline{\pi})\Delta\theta$. When the signal is informative, $\lambda>\frac{1}{2}$, we have $\bar{\pi}>\pi>\underline{\pi}$. The result in the lemma follows. QED

Proof of Lemma 2 Let $\mu_{\bar{s}}$ and $\mu_{\underline{s}}$ denote the Lagrange multipliers on the incentive compatibility constraints (6) and (7), respectively (μ again denotes the multiplier on the participation constraint (2)). The first-order conditions are

$$\begin{aligned}
\pi\lambda u'(\bar{\theta}+\tau(\bar{\theta},\bar{s}))-\mu_{\bar{s}}\bar{\pi}-\mu\pi\lambda &= 0\\
(1-\pi)(1-\lambda)u'(\underline{\theta}+\tau(\underline{\theta},\bar{s}))-\mu_{\bar{s}}(1-\bar{\pi})-\mu(1-\pi)(1-\lambda) &= 0\\
\pi(1-\lambda)u'(\bar{\theta}+\tau(\bar{\theta},\underline{s}))-\mu_{\underline{s}}\underline{\pi}-\mu\pi(1-\lambda) &= 0\\
(1-\pi)\lambda u'(\underline{\theta}+\tau(\underline{\theta},\underline{s}))-\mu_{\underline{s}}(1-\underline{\pi})-\mu(1-\pi)\lambda &= 0
\end{aligned}$$

They rewrite as

$$\begin{aligned}
\bar{u}'(\tau(\bar{\theta},\bar{s})) &= \mu + \mu_{\bar{s}}\frac{\bar{\pi}}{\pi\lambda} && (32)\\
\underline{u}'(\tau(\underline{\theta},\bar{s})) &= \mu + \mu_{\bar{s}}\frac{1-\bar{\pi}}{(1-\pi)(1-\lambda)} && (33)\\
\bar{u}'(\tau(\bar{\theta},\underline{s})) &= \mu + \mu_{\underline{s}}\frac{\underline{\pi}}{\pi(1-\lambda)} && (34)\\
\underline{u}'(\tau(\underline{\theta},\underline{s})) &= \mu + \mu_{\underline{s}}\frac{1-\underline{\pi}}{(1-\pi)\lambda} && (35)
\end{aligned}$$

The participation constraint must bind. Suppose not, i.e. $\mu = 0$. Then, (32) and (33) imply that $\mu_{\bar{s}} > 0$. Similarly, (34) and (35) imply that $\mu_{\underline{s}} > 0$. Both incentive constraints bind so that $A\mathcal{P} = \bar{\tau} = \underline{\tau}$. Since the participation constraint is slack, it must be that $0 > E[\tau] \equiv \text{prob}[\bar{s}]\bar{\tau} + \text{prob}[\underline{s}]\underline{\tau} = A\mathcal{P}\left(\text{prob}[\bar{s}] + \text{prob}[\underline{s}]\right) = A\mathcal{P}$, which contradicts $A\mathcal{P} > 0$. Hence, the participation constraint binds, $E[\tau] = 0$.

Proof of Lemma 3 First, it cannot be that both incentive constraints are slack since we assume that the first-best is not attainable, $A\mathcal{P} < (\pi - \underline{\pi})\Delta\theta$. It also cannot be that both constraints are binding since $\bar{\tau} = \underline{\tau} = A\mathcal{P} > 0$ contradicts $E[\tau] = 0$ (Lemma 2). We now show that it is the incentive constraint following a *bad* signal that is binding. Suppose not, so that $A\mathcal{P} = \bar{\tau} > 0 > \underline{\tau}$ where the last inequality follows from the binding participation constraint $E[\tau] = 0$ (Lemma 2). Then, $\mu_{\underline{s}} = 0$ and $\mu_{\bar{s}} \geq 0$ and equations (32) through (35) yield

$$\bar{u}'(\tau(\bar{\theta},\underline{s})) = \underline{u}'(\tau(\underline{\theta},\underline{s})) = \mu \leq \bar{u}'(\tau(\bar{\theta},\bar{s})) = \underline{u}'(\tau(\underline{\theta},\bar{s}))$$

Comparing the first with the third term and the second with the fourth term yields $\tau(\bar{\theta},\underline{s}) \geq \tau(\bar{\theta},\bar{s})$ and $\tau(\underline{\theta},\underline{s}) \geq \tau(\underline{\theta},\bar{s})$. Moreover, from (32)-(35) and

$$\frac{\bar{\pi}}{\pi\lambda} = \frac{1-\bar{\pi}}{(1-\pi)(1-\lambda)} = \frac{1}{\text{prob}[\bar{s}]} \text{ and } \frac{\underline{\pi}}{\pi(1-\lambda)} = \frac{1-\underline{\pi}}{(1-\pi)\lambda} = \frac{1}{\text{prob}[\underline{s}]},$$

it follows that there is full risk-sharing conditional on the signal, i.e., $\bar{u}'(\tau(\bar{\theta},\bar{s})) = \underline{u}'(\tau(\underline{\theta},\bar{s}))$ and $\bar{u}'(\tau(\bar{\theta},\underline{s})) = \underline{u}'(\tau(\underline{\theta},\underline{s}))$ and thus $\tau(\underline{\theta},\bar{s}) - \tau(\bar{\theta},\bar{s}) = \tau(\underline{\theta},\underline{s}) - \tau(\bar{\theta},\underline{s}) = \Delta\theta > 0$. Using $\tau(\underline{\theta},\tilde{s}) > \tau(\bar{\theta},\tilde{s})$ and $\bar{\pi} > \underline{\pi}$, we can write

$$\begin{aligned}
0 &< \bar{\tau} \equiv \bar{\pi}\tau(\bar{\theta},\bar{s}) + (1-\bar{\pi})\tau(\underline{\theta},\bar{s})\\
&< \underline{\pi}\tau(\bar{\theta},\bar{s}) + (1-\underline{\pi})\tau(\underline{\theta},\bar{s})\\
&\leq \underline{\pi}\tau(\bar{\theta},\underline{s}) + (1-\underline{\pi})\tau(\underline{\theta},\underline{s}) \equiv \underline{\tau}
\end{aligned}$$

But $\underline{\tau} < 0$, hence there is a contradiction. Consequently, only the incentive constraint after a bad signal binds. QED

Proof of Proposition 2 The optimal contract is given by the binding incentive constraint following a bad signal, $A\mathcal{P} = \underline{\tau}$ (Lemma 3), the binding participation constraint, $E(\tau) = 0$ (Lemma 2), and full risk-sharing conditional on the signal (10) (in the proof of Lemma 3). QED

Proof of Proposition 3 Let $\mu_{\bar{s}}$ and $\mu_{\underline{s}}$ denote the Lagrange multipliers on the incentive compatibility constraints (13) and (14), respectively, and let μ denote the multiplier on the participation constraint (15). The first-order conditions with respect to transfers $\tau(\bar{\theta}, \bar{s}, R)$, $\tau(\underline{\theta}, \bar{s}, R)$, $\tau(\bar{\theta}, \underline{s}, R)$, $\tau(\underline{\theta}, \underline{s}, R)$ and $\tau(\bar{\theta}, \underline{s}, 0)$ are:

$$\begin{aligned}
\bar{u}'(\tau(\bar{\theta}, \bar{s}, R)) &= \mu + \frac{\mu_{\bar{s}}}{\text{prob}[\bar{s}]} && (36)\\
\underline{u}'(\tau(\underline{\theta}, \bar{s}, R)) &= \mu + \frac{\mu_{\bar{s}}}{\text{prob}[\bar{s}]} && (37)\\
\bar{u}'(\tau(\bar{\theta}, \underline{s}, R)) &= \mu - \frac{\mu_{\underline{s}}}{p\text{prob}[\underline{s}]} && (38)\\
\underline{u}'(\tau(\underline{\theta}, \underline{s}, R)) &= \mu - \frac{\mu_{\underline{s}}}{p\text{prob}[\underline{s}]} && (39)\\
\bar{u}'(\tau(\bar{\theta}, \underline{s}, 0)) &= \mu + \frac{\mu_{\underline{s}}}{(1-p)\,\text{prob}[\underline{s}]} && (40)
\end{aligned}$$

where we use the shorthand $\bar{u}'(\tau(\bar{\theta}, \tilde{s}, \tilde{R}))$ and $\underline{u}'(\tau(\underline{\theta}, \tilde{s}, \tilde{R}))$ to denote marginal utility in state $\bar{\theta}$ and $\underline{\theta}$, respectively, conditional on the signal $\tilde{s}$ and the return $\tilde{R}$.

We first show that the participation constraint binds. Suppose instead that the constraint is slack, implying $\mu = 0$. Since $\mu_{\underline{s}} \geq 0$, equations (38) and (39) cannot hold. A contradiction.

Next, we show that the incentive constraint after a bad signal (14) is slack, implying $\mu_{\underline{s}} = 0$. Suppose that the constraint binds and $A\mathcal{P}+\underline{\pi}\tau(\bar{\theta}, \underline{s}, 0) = \underline{\pi}\tau(\bar{\theta}, \underline{s}, R)+(1-\underline{\pi})\tau(\underline{\theta}, \underline{s}, R)$ implying that

$$\underline{\pi}\tau(\bar{\theta}, \underline{s}, R) + (1-\underline{\pi})\tau(\underline{\theta}, \underline{s}, R) \leq A\mathcal{P} \tag{41}$$

since $\tau(\bar{\theta}, \underline{s}, 0) \leq 0$. The participation constraint binds and rewrites as

$$\begin{aligned}
-\,\text{prob}[\underline{s}](1-p)\left[A\mathcal{P}+\underline{\pi}\tau(\bar{\theta}, \underline{s}, 0)\right] = \text{prob}[\bar{s}]\left[\bar{\pi}\tau(\bar{\theta}, \bar{s}, R) + (1-\bar{\pi})\tau(\underline{\theta}, \bar{s}, R)\right] +\\
\text{prob}[\underline{s}]p\left[\underline{\pi}\tau(\bar{\theta}, \underline{s}, R) + (1-\underline{\pi})\tau(\underline{\theta}, \underline{s}, R)\right]
\end{aligned}$$

Using the binding incentive constraint (14) in the equation above and simplifying yields

$$\text{prob}[\bar{s}]\left[\bar{\pi}\tau(\bar{\theta},\bar{s},R)+(1-\bar{\pi})\tau(\underline{\theta},\bar{s},R)\right]+\text{prob}[\underline{s}]\left[\underline{\pi}\tau(\bar{\theta},\underline{s},R)+(1-\underline{\pi})\tau(\underline{\theta},\underline{s},R)\right]=0 \quad (42)$$

Equations (41) and (42) imply that the optimal transfers $\tau(\bar{\theta},\bar{s},R)$, $\tau(\underline{\theta},\bar{s},R)$, $\tau(\bar{\theta},\underline{s},R)$ and $\tau(\underline{\theta},\underline{s},R)$ satisfy the incentive-compatibility condition inducing effort after bad news (7) and the participation constraint (2) in the contract with effort after both signals. Hence, inducing effort after both signals is feasible with these transfers. We now show that, given these transfers, the expected utility of the contract with effort after both signals is strictly higher than the expected utility of the contract without effort after bad news, i.e.:

$$\begin{aligned}&\pi\lambda u(\bar{\theta}+\tau(\bar{\theta},\bar{s},R))+(1-\pi)(1-\lambda)u(\underline{\theta}+\tau(\underline{\theta},\bar{s},R))+\pi(1-\lambda)u(\bar{\theta}+\tau(\bar{\theta},\underline{s},R))\\&+(1-\pi)\lambda u(\underline{\theta}+\tau(\underline{\theta},\underline{s},R))-\pi\lambda u(\bar{\theta}+\tau(\bar{\theta},\bar{s},R))-(1-\pi)(1-\lambda)u(\underline{\theta}+\tau(\underline{\theta},\bar{s},R))\\&-\pi(1-\lambda)[pu(\bar{\theta}+\tau(\bar{\theta},\underline{s},R))+(1-p)u(\bar{\theta}+\tau(\bar{\theta},\underline{s},0))]\\&-(1-\pi)\lambda[pu(\underline{\theta}+\tau(\underline{\theta},\underline{s},R))+(1-p)u(\underline{\theta})]\geq 0\end{aligned}$$

The left-hand side simplifies to:

$$\pi(1-\lambda)(1-p)\left[u(\bar{\theta}+\tau(\bar{\theta},\underline{s},R))-u(\bar{\theta}+\tau(\bar{\theta},\underline{s},0))\right]+(1-\pi)\lambda(1-p)\left[u(\underline{\theta}+\tau(\underline{\theta},\underline{s},R))-u(\underline{\theta})\right]$$

It follows from equations (38) and (40) that $\bar{u}'(\tau(\bar{\theta},\underline{s},R))\leq\bar{u}'(\tau(\bar{\theta},\underline{s},0))$ and thus $\tau(\bar{\theta},\underline{s},R)\geq\tau(\bar{\theta},\underline{s},0)$. Hence, the expression in the first square bracket is non-negative. Next note that $\tau(\underline{\theta},\underline{s},R)>0$. Suppose not. Equations (36)-(40) imply that $0\geq\tau(\underline{\theta},\underline{s},R)\geq\tau(\underline{\theta},\bar{s},R)>\tau(\bar{\theta},\bar{s},R)$ and $0\geq\tau(\underline{\theta},\underline{s},R)>\tau(\bar{\theta},\underline{s},R)\geq\tau(\bar{\theta},\underline{s},0)$. But optimal transfers cannot be all zero or negative as the buyer would then get no insurance. Since $\tau(\underline{\theta},\underline{s},R)>0$ the expression in the second square bracket is positive. Hence, the protection buyer prefers to induce effort after bad news, contradicting the optimality of the contract with risk-taking after bad news. We conclude that if risk-taking after bad news is optimal, the incentive constraint after a bad signal (14) must be slack.

Hence, we have full sharing of the $\tilde{\theta}$ risk conditional on the signal except when the seller defaults in the $\underline{\theta}$ state: $\bar{u}'(\tau(\bar{\theta},\bar{s},R))=\underline{u}'(\tau(\underline{\theta},\bar{s},R))$ and $\bar{u}'(\tau(\bar{\theta},\underline{s},R))=\underline{u}'(\tau(\underline{\theta},s,R))=\bar{u}'(\tau(\bar{\theta},\underline{s},0))$. Therefore

$$\tau(\underline{\theta},\tilde{s})-\tau(\bar{\theta},\tilde{s})=\Delta\theta>0 \quad (43)$$

We now show that the incentive constraint after a good signal (13) is also slack, implying $\mu_{\bar{s}}=0$. When the constraint is slack, there is full insurance except when the seller defaults in $\underline{\theta}$ state, i.e. we have:

$$\tau(\tilde{\theta}, \bar{s}, R) = \tau(\tilde{\theta}, \underline{s}, R) \text{ and } \tau(\bar{\theta}, \underline{s}, R) = \tau(\bar{\theta}, \underline{s}, 0) \tag{44}$$

The optimal contract in this case is given by equations (43), (44) and the binding participation constraint. We now check under what conditions the incentive constraint following a good signal is indeed slack with that contract. Starting with the binding participation constraint and using (43) and (44), we get

$$\begin{aligned}
-\operatorname{prob}[\underline{s}](1-p)A\mathcal{P} = {} & \operatorname{prob}[\bar{s}][\tau(\underline{\theta}, \underline{s}, R) - \bar{\pi}\Delta\theta] \\
& + \operatorname{prob}[\underline{s}]p[\tau(\underline{\theta}, \underline{s}, R) - \underline{\pi}\Delta\theta] + (1-p)\operatorname{prob}[\underline{s}]\underline{\pi}\left[\tau(\underline{\theta}, \underline{s}, R) - \Delta\theta\right]
\end{aligned}$$

Hence,

$$\tau(\underline{\theta}, \underline{s}, R) = \frac{\pi\Delta\theta - \operatorname{prob}[\underline{s}]\,(1-p)\,A\mathcal{P}}{1 - \operatorname{prob}[\underline{s}]\,(1-\underline{\pi})\,(1-p)} \tag{45}$$

For the incentive constraint following a good signal (13) to be slack, it must be that

$$A\mathcal{P} > \bar{\pi}\tau(\bar{\theta}, \bar{s}, R) + (1-\bar{\pi})\tau(\underline{\theta}, \bar{s}, R) = \tau(\underline{\theta}, \underline{s}, R) - \bar{\pi}\Delta\theta$$

or, after substituting for $\tau(\underline{\theta}, \underline{s})$ and simplifying,

$$A\mathcal{P} > \Delta\theta \frac{\pi - \bar{\pi}\left[1 - \operatorname{prob}[\underline{s}]\,(1-\underline{\pi})\,(1-p)\right]}{1 + \operatorname{prob}[\underline{s}]\underline{\pi}\,(1-p)} \tag{46}$$

Condition (46) is always satisfied if

$$\pi - \bar{\pi}\left[1 - \operatorname{prob}[\underline{s}]\,(1-\underline{\pi})\,(1-p)\right] < 0 \tag{47}$$

since $A\mathcal{P} > 0$. Condition (47) is equivalent to $\lambda^2(1-p) - 2\lambda + 1 < 0$. This inequality holds under our assumption (9), i.e. for all $\lambda \geq \lambda^* \equiv \frac{1-\sqrt{p}}{1-p} > \frac{1}{2}$. This is because the left-hand side of the inequality above is decreasing in λ and it is equal to zero for λ^*. QED

Proof of Proposition 4 The proof proceeds in three steps. First, we show that the expected utility of the protection buyer when the protection seller is exerting effort after both signals is increasing in $\mathcal{P}$. Its derivative is

$$\begin{aligned}
& -\frac{\operatorname{prob}[\underline{s}]A}{\operatorname{prob}[\bar{s}]}\left[\pi\lambda\bar{u}'(\tau(\bar{\theta}, \bar{s})) + (1-\pi)\,(1-\lambda)\,\underline{u}'(\tau(\underline{\theta}, \bar{s}))\right] \\
& + \pi\,(1-\lambda)\,\bar{u}'(\tau(\bar{\theta}, \underline{s})) + (1-\pi)\,\lambda\underline{u}'(\tau(\underline{\theta}, \underline{s})) \\
& = \operatorname{prob}[\underline{s}]A\left[\bar{u}'(\tau(\bar{\theta}, \underline{s})) - \bar{u}'(\tau(\bar{\theta}, \bar{s}))\right] > 0
\end{aligned}$$

since $\tau(\bar{\theta}, \underline{s}) < \tau(\bar{\theta}, \bar{s})$.

Second, we show that the expected utility of the protection buyer when there is risk-taking after bad news is decreasing in $\mathcal{P}$. Its derivative is

$$\begin{aligned}
& -\frac{\text{prob}[\underline{s}]\,(1-p)\,A}{1-\text{prob}[\underline{s}]\,(1-\underline{\pi})\,(1-p)}\left[\pi\lambda\bar{u}'(\tau(\bar{\theta},\bar{s})) + (1-\pi)\,(1-\lambda)\,\underline{u}'(\tau(\underline{\theta},\bar{s}))\right.\\
& \left. + \pi\,(1-\lambda)\,\bar{u}'(\tau(\bar{\theta},\underline{s})) + (1-\pi)\,\lambda p\underline{u}'(\tau(\underline{\theta},\underline{s}))\right]\\
& = -\frac{\text{prob}[\underline{s}]\,(1-p)\,A}{1-\text{prob}[\underline{s}]\,(1-\underline{\pi})\,(1-p)}\left[\pi\bar{u}'(\tau(\bar{\theta},\bar{s})) + (1-\pi)((1-\lambda)+p\lambda)\underline{u}'(\tau(\underline{\theta},\bar{s}))\right] < 0
\end{aligned}$$

Third, we provide a sufficient condition under which, when $\mathcal{P} = 0$, the expected utility of the protection buyer is smaller when effort is requested after bad news than when it is not. The expected utility of the protection buyer when effort is requested after bad news is

$$\begin{aligned}
& [\pi\lambda + (1-\pi)\,(1-\lambda)]\,u\,(\underline{\theta} + \bar{\pi}\Delta\theta) + [\pi\,(1-\lambda) + (1-\pi)\,\lambda]\,u\,(\underline{\theta} + \underline{\pi}\Delta\theta)\\
& = \text{prob}[\bar{s}]u\,(\underline{\theta} + \bar{\pi}\Delta\theta) + \text{prob}[\underline{s}]u\,(\underline{\theta} + \underline{\pi}\Delta\theta)\\
& = \text{prob}[\bar{s}]u(E[\tilde{\theta}|\bar{s}]) + \text{prob}[\underline{s}]u(E[\tilde{\theta}|\underline{s}])
\end{aligned} \tag{48}$$

With risk-taking after bad news it is

$$\begin{aligned}
& (\text{prob}[\bar{s}] + \text{prob}[\underline{s}]\,(p + (1-p)\,\underline{\pi}))\,u\left(\underline{\theta} + \frac{\pi\Delta\theta}{1-\text{prob}[\underline{s}]\,(1-\underline{\pi})\,(1-p)}\right)\\
& + (1-p)\,\text{prob}[\underline{s}]\,(1-\underline{\pi})\,u(\underline{\theta})\\
& = (\text{prob}[\bar{s}] + p\text{prob}[\underline{s}])\,u\left(\hat{E}[\tilde{\theta}]\right) + (1-p)\,\text{prob}[\underline{s}]\left(\underline{\pi}u\left(\hat{E}[\tilde{\theta}]\right) + (1-\underline{\pi})\,u(\underline{\theta})\right)
\end{aligned} \tag{49}$$

where $\hat{E}[\tilde{\theta}] \equiv \hat{\pi}\bar{\theta} + (1-\hat{\pi})\underline{\theta}$ and $\hat{\pi} \equiv \frac{\pi}{1-\text{prob}[\underline{s}](1-\underline{\pi})(1-p)}$. Note that

$$\bar{\pi} > \hat{\pi} > \pi > \underline{\pi} \tag{50}$$

for $p \in (0,1)$. The last two inequalities are straightforward. The first inequality holds if and only if $\lambda^2(1-p) - 2\lambda + 1 < 0$ which is satisfied under our assumption (9), i.e. for all $\lambda \geq \lambda^* \equiv \frac{1-\sqrt{p}}{1-p} > \frac{1}{2}$.

Combining (48) and (49), we have that no effort after a bad signal dominates effort (when $\mathcal{P} = 0$) if and only if

$$\begin{aligned}
& \text{prob}[\bar{s}]u(E[\tilde{\theta}|\bar{s}]) + \text{prob}[\underline{s}]u(E[\tilde{\theta}|\underline{s}])\\
& < (\text{prob}[\bar{s}] + \text{prob}[\underline{s}]p)\,u(\hat{E}[\tilde{\theta}]) + \text{prob}[\underline{s}](1-p)EU(\tilde{R} = 0)
\end{aligned}$$

where $EU(\tilde{R}=0) \equiv \underline{\pi} u(\hat{E}[\tilde{\theta}]) + (1-\underline{\pi})u(\underline{\theta})$. After collecting terms, we have

$$\begin{aligned} &\text{prob}[\bar{s}]\left[u(E[\tilde{\theta}|\bar{s}]) - u(\hat{E}[\tilde{\theta}])\right] + \text{prob}[\underline{s}]\left[u(E[\tilde{\theta}|\underline{s}]) - EU(\tilde{R}=0)\right] \\ &< p\text{prob}[\underline{s}]\left[u(\hat{E}[\tilde{\theta}]) - EU(\tilde{R}=0)\right] \end{aligned}$$

All the differences in the square brackets are positive. The first one due to (50), the second one due to the concavity of u, and the third one due to both the concavity of u and (50).

Rearranging, we arrive at

$$\frac{\text{prob}[\bar{s}]}{\text{prob}[\underline{s}]} \frac{u(E[\tilde{\theta}|\bar{s}]) - u(\hat{E}[\tilde{\theta}])}{u(\hat{E}[\tilde{\theta}]) - EU(\tilde{R}=0)} + \frac{u(E[\tilde{\theta}|\underline{s}]) - EU(\tilde{R}=0)}{u(\hat{E}[\tilde{\theta}]) - EU(\tilde{R}=0)} < p \tag{51}$$

It is clear that the left-hand side is strictly positive so that seller's effort dominates when p is small. The left-hand is, however, also strictly smaller than one so that no effort after a bad signal dominates when p is high.[15]

The condition

$$\frac{\text{prob}[\bar{s}]}{\text{prob}[\underline{s}]} \frac{u(E[\tilde{\theta}|\bar{s}]) - u(\hat{E}[\tilde{\theta}])}{u(\hat{E}[\tilde{\theta}]) - EU(\tilde{R}=0)} + \frac{u(E[\tilde{\theta}|\underline{s}]) - EU(\tilde{R}=0)}{u(\hat{E}[\tilde{\theta}]) - EU(\tilde{R}=0)} < 1$$

simplifies to $\text{prob}[\bar{s}]u(E[\tilde{\theta}|\bar{s}]) + \text{prob}[\underline{s}]u(E[\tilde{\theta}|\underline{s}]) < u(\hat{E}[\tilde{\theta}])$. By concavity,

$$prob[\bar{s}]u(E[\tilde{\theta}|\bar{s}]) + prob[\underline{s}]u(E[\tilde{\theta}|\underline{s}]) < u(E[\tilde{\theta}])$$

and so the condition holds when $u(E[\tilde{\theta}]) \leq u(\hat{E}[\tilde{\theta}])$ which is always true due to (50).

If it is optimal to have risk-taking after bad news when the protection seller has zero pledgeable income, it remains so for low levels of $\mathcal{P}$. There is a level of per-unit pledgeable income such that the expected utility of the protection buyer when effort is requested after bad news is just equal to its counterpart with risk-taking after bad news. Denote it by $\hat{\mathcal{P}}$. For $\mathcal{P} \geq \hat{\mathcal{P}}$ the optimal contract requests effort after good and bad news. QED

Proof of Lemma 4 Let μ and $\mu_{\underline{s}}$ denote the Lagrange multipliers on the participation and incentive-compatibility constraints (21) and (22), respectively. Let μ_0 and μ_1 be the Lagrange multipliers on the feasibility constraints $\alpha \geq 0$ and $\alpha \leq 1$, respectively. The first-order conditions with respect to expected transfers $\bar{\tau}$, $\underline{\tau}$ and the margin α are:

$$u'(E[\theta|\bar{s}] + \bar{\tau}) = \mu \tag{52}$$

$$u'(E[\theta|\underline{s}] + \underline{\tau}) = \mu + \frac{\mu_{\underline{s}}}{\text{prob}[\underline{s}]} \tag{53}$$

$$\mu_{\underline{s}} A(1-\mathcal{P}) + \mu_0 = \mu \text{prob}[\underline{s}] A(R-1) + \mu_1 \tag{54}$$

[15]Note that this inequality is evaluated at $\mathcal{P}=0$ and $\mathcal{P}$ is a function of p. There is, however, an open set of parameters for which no effort after a bad signal dominates.

where $u'(E[\theta|\bar{s}]+\bar{\tau})$ and $u'(E[\theta|\underline{s}]+\underline{\tau})$ are marginal utilities conditional on the good and the bad signal, respectively. Equation (52) implies that $\mu > 0$, i.e., the participation constraint binds. Substituting (52) and (53) into (54), yields

$$\frac{u'(E[\theta|\underline{s}]+\underline{\tau})}{u'(E[\theta|\bar{s}]+\bar{\tau})} = 1 + \frac{R-1}{1-\mathcal{P}} + \frac{\mu_1 - \mu_0}{u'(E[\theta|\bar{s}]+\bar{\tau})\text{prob}[\underline{s}]\,(1-\mathcal{P})\,A} \tag{55}$$

For any optimal $\alpha \in [0,1]$, $\mu_{\underline{s}} > 0$ and the incentive constraint after bad news is binding. This is because of the following: For $\alpha = 0$, we are solving the same problem as in Section 4.1 and the claim follows from Lemma 3. For $0 < \alpha \leq 1$, we have $\mu_0 = 0$ and equation (55) implies that $\frac{u'(E[\theta|\underline{s}]+\underline{\tau})}{u'(E[\theta|\bar{s}]+\bar{\tau})} > 1$. But then, by equations (52) and (53), it must be that $\mu_{\underline{s}} > 0$. QED

Proof of Lemma 5 Consider equation (54) in the proof of Lemma 4. Since $\mu_1 \geq 0$, the right-hand side is strictly positive. Now, suppose $\mathcal{P} \geq 1$. Then, it must be that $\mu_0 > 0$ for equation (54) to hold. Hence, $\alpha^* = 0$ and margins are not used for $\mathcal{P} \geq 1$. QED

Proof of Proposition 5 Recall that φ is decreasing in α. Hence, if $\varphi(0) < 1 + \frac{R-1}{1-\mathcal{P}}$, then $\varphi(\alpha) < 1 + \frac{R-1}{1-\mathcal{P}}$ for any $\alpha \in [0,1]$. From equation (54) in the proof of Lemma 4 we then have have $\mu_0 > 0$ and hence $\alpha^* = 0$. By the same logic, if $\varphi(1) > 1 + \frac{R-1}{1-\mathcal{P}}$, then $\mu_1 > 0$ and hence $\alpha^* = 1$. Otherwise, $\alpha^* \in (0,1)$ is given by $\varphi(\alpha^*) = 1 + \frac{R-1}{1-\mathcal{P}}$. QED

Proof of Lemma 6 Let $\mu_{\bar{s}}$ and $\mu_{\underline{s}}$ denote the Lagrange multipliers on the incentive compatibility constraints (13) and (28), respectively, and let μ denote the multiplier on the participation constraint (27). Furthermore, let μ_0 and μ_1 be the Lagrange multipliers on the feasibility constraints $\alpha \geq 0$ and $\alpha \leq 1$, and let μ_2 and μ_3 be the Lagrange multipliers on the constraints $\alpha A \geq \tau(\underline{\theta}, \underline{s}, 0)$ and $\alpha A \geq \tau(\bar{\theta}, \underline{s}, 0)$, respectively. The first-order conditions with respect to transfers $\tau(\bar{\theta}, \bar{s}, R)$, $\tau(\underline{\theta}, \bar{s}, R)$, $\tau(\bar{\theta}, \underline{s}, R)$, $\tau(\underline{\theta}, \underline{s}, R)$, $\tau(\bar{\theta}, \underline{s}, 0)$, $\tau(\underline{\theta}, \underline{s}, 0)$, and α are:

$$\bar{u}'(\tau(\bar{\theta}, \bar{s}, R)) = \mu + \frac{\mu_{\bar{s}}}{\text{prob}[\bar{s}]} \tag{56}$$

$$\underline{u}'(\tau(\underline{\theta}, \bar{s}, R)) = \mu + \frac{\mu_{\bar{s}}}{\text{prob}[\bar{s}]} \tag{57}$$

$$\bar{u}'(\tau(\bar{\theta}, \underline{s}, R)) = \mu - \frac{\mu_{\underline{s}}}{p\text{prob}[\underline{s}]} \tag{58}$$

$$\underline{u}'(\tau(\underline{\theta}, \underline{s}, R)) = \mu - \frac{\mu_{\underline{s}}}{p\text{prob}[\underline{s}]} \tag{59}$$

$$\bar{u}'(\tau(\bar{\theta}, \underline{s}, 0)) = \mu + \frac{\mu_{\underline{s}}}{(1-p)\,\text{prob}[\underline{s}]} + \frac{\mu_3}{(1-\lambda)\,\pi\,(1-p)} \tag{60}$$

$$\underline{u}'(\tau(\underline{\theta}, \underline{s}, 0)) = \mu + \frac{\mu_{\underline{s}}}{(1-p)\,\text{prob}[\underline{s}]} + \frac{\mu_2}{(1-\pi)\,\lambda\,(1-p)} \tag{61}$$

$$\mu \text{prob}[\underline{s}] A \left[R - 1 - (1-p)\,\mathcal{P}\right] + \mu_1 = \mu_0 + A\mathcal{P}\mu_{\underline{s}} + A\left(\mu_2 + \mu_3\right) \tag{62}$$

where we use the shorthand $\bar{u}'(\tau(\bar{\theta}, \tilde{s}, \tilde{R}))$ and $\underline{u}'(\tau(\underline{\theta}, \tilde{s}, \tilde{R}))$ to denote marginal utility in state $\bar{\theta}$ and $\underline{\theta}$ conditional on the signal $\tilde{s}$ and the return $\tilde{R}$. We must have that $\mu > 0$, i.e., the participation constraint binds. Otherwise, equations (58) and (59) cannot hold. QED

Proof of Lemma 7 First, note that if $pR + B < 1$ or, equivalently, $R - 1 < (1-p)\,\mathcal{P}$, it follows from equation (62) in the proof of Lemma 6 that $\mu_1 > 0$ must hold since the right-hand side of (62) is non-negative. Hence, $\alpha^* = 1$. Second, we claim that if $\alpha^* = 1$, risk-taking after bad news cannot be strictly optimal. For $\alpha^* = 1$, the entire balance sheet of the protection seller is put in the margin after bad news. It is thus ring-fenced from moral hazard, making effort and no effort equivalent. But then, the contract with effort dominates the contract without effort (and strictly so when the optimal contract with effort has $\alpha < 1$). QED

Proof of Lemma 8 We know from the proof of Lemma 6 that the participation constraint binds. We also have that $\alpha^* < 1$ (and $\mu_1 = 0$) since if $\alpha^* = 1$, risk-taking cannot be strictly optimal.

First, we show that the incentive constraint after bad news must be slack and $\mu_{\underline{s}} = 0$. Suppose otherwise. Then, we have

$$\underline{\pi}\tau(\bar{\theta}, \underline{s}, R) + (1-\underline{\pi})\tau(\underline{\theta}, \underline{s}, R) = (1-\alpha)\,A\mathcal{P} + \underline{\pi}\tau(\bar{\theta}, \underline{s}, 0) + (1-\underline{\pi})\tau(\underline{\theta}, \underline{s}, 0) \tag{63}$$

Since $\tau(\bar{\theta}, \underline{s}, 0) \leq \alpha A$ and $\tau(\underline{\theta}, \underline{s}, 0) \leq \alpha A$, we have $\underline{\pi}\tau(\bar{\theta}, \underline{s}, R) + (1-\underline{\pi})\tau(\underline{\theta}, \underline{s}, R) \leq (1-\alpha)\,A\mathcal{P} + \alpha A$ implying that transfers $\tau(\bar{\theta}, \underline{s}, R)$ and $\tau(\underline{\theta}, \underline{s}, R)$ satisfy the incentive-compatibility condition that induces effort after bad news (22). Using the binding participation constraint, we get

$$\begin{aligned} &-\text{prob}[\underline{s}]\left[\alpha A\,(R-1) + (1-p)\left((1-\alpha)A\mathcal{P} + \underline{\pi}\tau(\bar{\theta}, \underline{s}, 0) + (1-\underline{\pi})\tau(\underline{\theta}, \underline{s}, 0)\right)\right] \\ &= \text{prob}[\bar{s}]\left[\bar{\pi}\tau(\bar{\theta}, \bar{s}, R) + (1-\bar{\pi})\tau(\underline{\theta}, \bar{s}, R)\right] + \text{prob}[\underline{s}]p\left[\underline{\pi}\tau(\bar{\theta}, \underline{s}, R) + (1-\underline{\pi})\tau(\underline{\theta}, \underline{s}, R)\right] \end{aligned}$$

or, using (63) and simplifying,

$$-\operatorname{prob}[\underline{s}]\alpha A\,(R-1) = \operatorname{prob}[\bar{s}]\left[\bar{\pi}\tau(\bar{\theta},\bar{s},R) + (1-\bar{\pi})\tau(\underline{\theta},\bar{s},R)\right] + \operatorname{prob}[\underline{s}]\left[\underline{\pi}\tau(\bar{\theta},\underline{s},R) + (1-\underline{\pi})\tau(\underline{\theta},\underline{s},R)\right]$$

so that the transfers in the equation above satisfy the participation constraint in the contract with margins and effort after both signals. Hence, inducing effort after both signals is feasible with these transfers. Using the same steps as in the proof of Proposition 3, we show that the expected utility of the contract with margins and effort after both signals is at least as high as the expected utility of the contract with margins and risk-taking after bad news, contradicting the optimality of the contract with risk-taking after bad news. Showing this is equivalent to showing that

$$\pi(1-\lambda)\left[u(\bar{\theta}+\tau(\bar{\theta},\underline{s},R)) - u(\bar{\theta}+\tau(\bar{\theta},\underline{s},0))\right] + (1-\pi)\lambda\left[u(\underline{\theta}+\tau(\underline{\theta},\underline{s},R)) - u(\underline{\theta}+\tau(\underline{\theta},\underline{s},0))\right]$$

is non-negative. The expression in the first square bracket is non-negative by (58) and (60). The expression in the second square bracket is non-negative by (59) and (61). This completes the proof of the claim.

Second, we prove by contradiction that $\alpha A > \tau(\bar{\theta},\underline{s},0)$ (and $\mu_3 = 0$). Hence, suppose that $\alpha A = \tau(\bar{\theta},\underline{s},0)$ and $\mu_3 \geq 0$. Given the feasibility constraint on $\tau(\underline{\theta},\underline{s},0)$, we can either have $\alpha A = \tau(\bar{\theta},\underline{s},0) > \tau(\underline{\theta},\underline{s},0)$, or $\alpha A = \tau(\bar{\theta},\underline{s},0) = \tau(\underline{\theta},\underline{s},0)$. We first show that $\tau(\bar{\theta},\underline{s},0) > \tau(\underline{\theta},\underline{s},0)$ cannot hold. Suppose otherwise, so that $\mu_2 = 0$. Equations (60) and (61) imply that $\bar{u}'(\tau(\bar{\theta},\underline{s},0)) \geq \underline{u}'(\tau(\underline{\theta},\underline{s},0))$ so that $\bar{\theta}+\tau(\bar{\theta},\underline{s},0) \leq \underline{\theta}+\tau(\underline{\theta},\underline{s},0)$ or, equivalently, $\Delta\theta + \alpha A \leq \tau(\underline{\theta},\underline{s},0)$ which contradicts $\tau(\underline{\theta},\underline{s},0) < \alpha A$. We next show that $\tau(\bar{\theta},\underline{s},0) = \tau(\underline{\theta},\underline{s},0)$ cannot hold either. Suppose otherwise, so that $\alpha A = \tau(\bar{\theta},\underline{s},0) = \tau(\underline{\theta},\underline{s},0)$. By (58) and (60), $\tau(\bar{\theta},\underline{s},R) \geq \tau(\bar{\theta},\underline{s},0)$, and by (59) and (61), $\tau(\underline{\theta},\underline{s},R) \geq \tau(\underline{\theta},\underline{s},0)$ implying

$$\begin{aligned} \tau(\bar{\theta},\underline{s},R) &\geq \tau(\bar{\theta},\underline{s},0) = \alpha A \geq 0 \\ \tau(\underline{\theta},\underline{s},R) &\geq \tau(\underline{\theta},\underline{s},0) = \alpha A \geq 0 \end{aligned} \tag{64}$$

Since the participation constraint binds, we have

$$\begin{aligned} &\operatorname{prob}[\bar{s}]\left[\bar{\pi}\tau(\bar{\theta},\bar{s},R) + (1-\bar{\pi})\tau(\underline{\theta},\bar{s},R)\right] + \operatorname{prob}[\underline{s}]p\left[\underline{\pi}\tau(\bar{\theta},\underline{s},R) + (1-\underline{\pi})\tau(\underline{\theta},\underline{s},R)\right] \\ &= -\operatorname{prob}[\underline{s}]\left[\alpha A\,(R-1) + (1-p)\,(\alpha A + (1-\alpha)A\mathcal{P})\right] \end{aligned}$$

The right-hand side of the expression above is negative. On the left-hand side, the second term in square bracket is non-negative by (64). We now show that the first term in square bracket is also non-negative, implying that the equation above cannot hold. We either have that the incentive constraint after good news binds or it is slack. If it binds, then the first term on the left-hand side (in square bracket) is equal to $A\mathcal{P} > 0$. If it is slack, then $\mu_{\bar{s}} = 0$ and $\tau(\bar{\theta}, \bar{s}, R) = \tau(\bar{\theta}, \underline{s}, R) \geq \alpha A$ while $\tau(\underline{\theta}, \bar{s}, R) = \tau(\underline{\theta}, \underline{s}, R) \geq \alpha A$ so that the expected transfer is non-negative. This completes the proof the claim.

Third, we show that $\alpha A = \tau(\underline{\theta}, \underline{s}, 0)$. Suppose otherwise, $\alpha A > \tau(\underline{\theta}, \underline{s}, 0)$ (i.e., $\mu_2 = 0$). Then, by equations (58) through (61), there is full insurance conditional on bad news, and expected transfers after bad news are equal to zero. But then, the incentive constraint after bad news (28) cannot be slack for any $\alpha \in [0, 1]$. A contradiction.

4) Fourth, we solve the relaxed problem, where the incentive constraint after good news is not imposed (and $\mu_{\bar{s}} = 0$). Then, we verify that the constraint holds for the solution of the relaxed problem. Equation (62) yields

$$0 \leq \mu_2 = \mu \text{prob}[\underline{s}] \left[R - 1 - (1-p)\,\mathcal{P}\right] - \frac{\mu_0}{A}$$

Replacing $R - 1 - (1-p)\,\mathcal{P}$ with $pR + B - 1$, and using the first-order conditions to substitute for μ and μ_2, we arrive at:

$$\frac{\underline{u}'(\tau(\underline{\theta}, \underline{s}, 0))}{\underline{u}'(\tau(\underline{\theta}, \underline{s}, R))} \equiv \phi(\alpha) = 1 + \frac{pR + B - 1}{(1-p)\,(1-\underline{\pi})} - \frac{\mu_0}{\underline{u}'(\tau(\underline{\theta}, \underline{s}, R))A\,(1-p)\,\text{prob}[\underline{s}]\,(1-\underline{\pi})} \quad (65)$$

For $\mu_{\underline{s}} = \mu_{\bar{s}} = \mu_3 = 0$, we have by (56) through (60) that $\tau(\bar{\theta}, \bar{s}, R) = \tau(\bar{\theta}, \underline{s}, R) = \tau(\bar{\theta}, \underline{s}, 0)$ and $\tau(\underline{\theta}, \bar{s}, R) = \tau(\underline{\theta}, \underline{s}, R)$. Using the participation constraint, this gives the optimal transfers stated in the lemma.

For this candidate contract, the incentive constraint (13) is slack if and only if

$$A\mathcal{P} + \bar{\pi}\Delta\theta > \frac{\underline{\pi}\Delta\theta - \text{prob}[\underline{s}]\,(1-p)\,A\mathcal{P}}{1 - \text{prob}[\underline{s}]\,(1-\underline{\pi})\,(1-p)} - \alpha A \frac{\text{prob}[\underline{s}]\,[pR + B - 1 + (1-\underline{\pi})\,(1-p)]}{1 - \text{prob}[\underline{s}]\,(1-\underline{\pi})\,(1-p)},$$

which holds under (9). QED

Proof of Proposition 6 Recall that ϕ is decreasing in α. Hence, if $\phi(0) < 1 + \frac{pR+B-1}{(1-p)(1-\underline{\pi})}$, then $\phi(\alpha) < 1 + \frac{pR+B-1}{(1-p)(1-\underline{\pi})}$ for any $\alpha \in [0, 1]$. So we must have $\mu_0 > 0$ and hence margins are not used. If $\phi(1) > 1 + \frac{pR+B-1}{(1-p)(1-\underline{\pi})}$, then (65) cannot hold for any $\alpha \in [0, 1)$ and the contract with margins and no effort cannot be strictly optimal. Otherwise, the optimal margin is as given in (30). QED

Proof of Proposition 7 Consider the contract with effort after both signals. Suppose, contrary to the claim in the proposition, that there exists a contract $(\bar{\tau}_i', \underline{\tau}_i') \neq (\bar{\tau}_i, \underline{\tau}_i)$, $i = 1, \ldots, N$, which satisfies participation and incentive constraints of each protection seller and yields a higher utility for the protection buyer.

Since $\bar{\tau}_i' \leq A\mathcal{P}_i$ and $\underline{\tau}_i' \leq A\mathcal{P}_i$ with $\mathcal{P}_i = \frac{\mathcal{P}}{N}$ holds for each i, we have $\sum_{i=1}^{N} \bar{\tau}_i' \leq A\mathcal{P}$ and $\sum_{i=1}^{N} \underline{\tau}_i' \leq A\mathcal{P}$. Similarly, $E[\tau_i'] \leq 0$ for each i implies that $\sum_{i=1}^{N} E[\tau_i'] \leq 0$. Let $\sum_{i=1}^{N} \bar{\tau}_i' \equiv \bar{\tau}'$ and $\sum_{i=1}^{N} \underline{\tau}_i' \equiv \underline{\tau}'$. Then, we have that

$$\text{prob}[\bar{s}]u(E[\tilde{\theta}|\bar{s}] + \bar{\tau}') + \text{prob}[\underline{s}]u(E[\tilde{\theta}|\underline{s}] + \underline{\tau}') > \text{prob}[\bar{s}]u(E[\tilde{\theta}|\bar{s}] + \bar{\tau}) + \text{prob}[\underline{s}]u(E[\tilde{\theta}|\underline{s}] + \underline{\tau})$$

But this contradicts the optimality of $\bar{\tau}$ and $\underline{\tau}$ for $N = 1$. QED

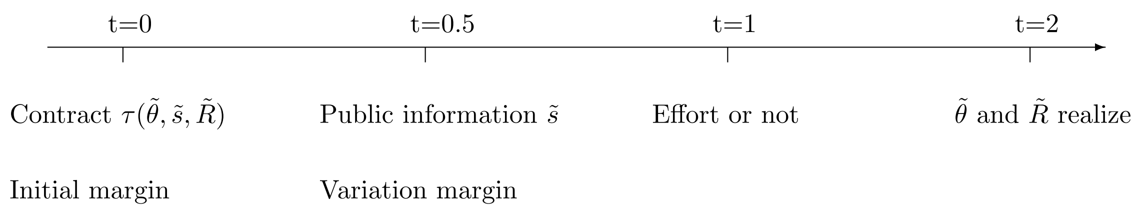

Figure 1: Sequence of events

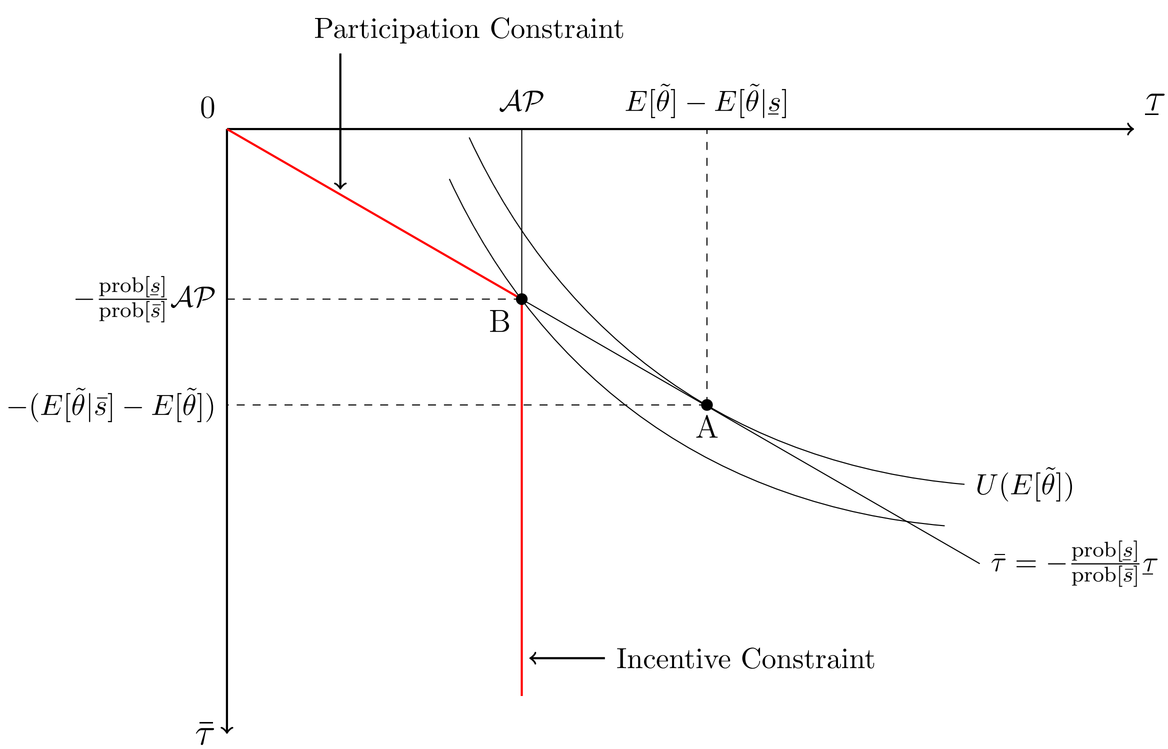

Figure 2: The figure shows the contract space with expected transfers conditional on the bad and the good signal on the x- and y-axis, respectively. Point A represents the first-best contract. Point B represents the optimal contract when the protection seller exerts risk-management effort after a bad signal.

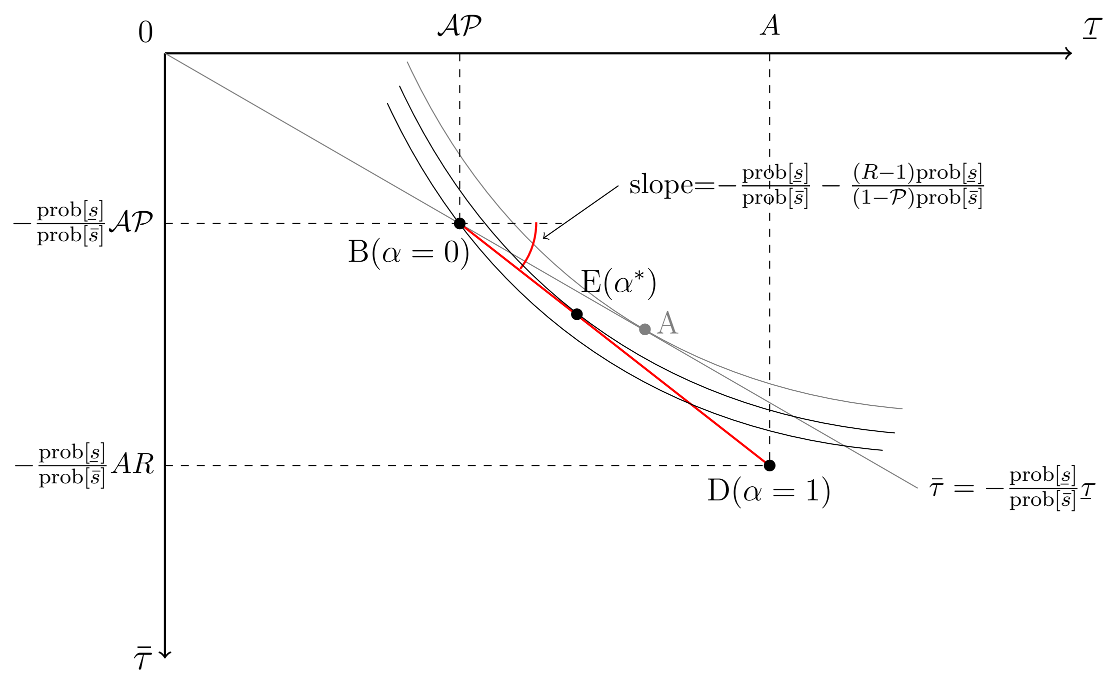

Figure 3: The figure shows the contract space with expected transfers conditional on the bad and the good signal on the x- and y-axis, respectively. Point E represents the optimal contract with margins when the protection seller exerts risk-management effort after a bad signal.